Chapter 2 - Electrostatics

Principle of Superposition

The interaction between two charges is completely unaffected by other charges

As a result, the total force on a particle and the potential are given by the sum of the two body forces and potentials respectively

Coulomb's Law and Electric Field

Force on charge Q at position $\vec{r}$ by collection of point charges q_j at $\vec{r}_j$

$\vec{F} = Q\vec{E}$ $\qquad\qquad$ $\vec{\mathscr{r}}_j = \vec{r} - \vec{r}_j$

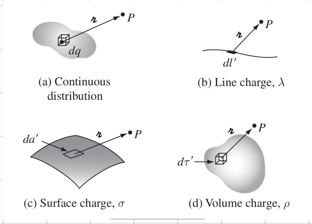
$\vec{E}(\vec{r}) = (4\pi\varepsilon_0)^{-1}\sum_j (q_j/\mathscr{r}_j^2)\hat{\mathscr{r}}_j$ $\qquad$ $\varepsilon_0 = 8.85\cdot 10^{-12}\ C^2N^{-1}m^{-1}$ $\qquad$ $\varepsilon_0 \equiv$ Permittivity of free space

Force on test charge Q position $\vec{r}$ due to continuous charge distribution

$\vec{F} = Q\vec{E}$

$\vec{E}(\vec{r}) = (4\pi\varepsilon_0)^{-1}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\,dq$ $\qquad$ $dq \longrightarrow \lambda\,dl \sim \sigma\,da \sim \rho\,d\tau$

$\hat{\mathscr{r}}$ is not constant and cannot come out as it changes depending on the position of the distribution treated



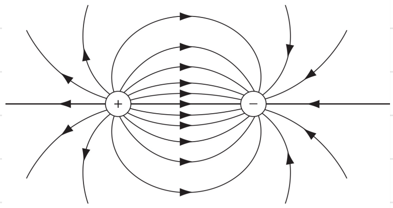
Field Lines and Flux

Field lines of describe strength and direction of field.

- The stronger the field, the denser the field lines i.e. bigger flux
- The further from source we are the rarer field lines get i.e. smaller flux due to Area $\propto r^2$

Any field line must either pass through the surface or start/terminate on an opposite charge

i.e. Flux is a measure of the total charge inside



Opposite charges

Flux of $\vec{E}$ through a surface S is given by $\Phi_E = \int_S \vec{E}\cdot d\vec{a}$

The flux obeys the superposition principle for all charges q_i enclosed in S

$\Phi_E = \sum_i \Phi_{E_i}$ where $\vec{E}_i$ is the electric field generated by a charge q_i

The total flux is a collection of the flux through S of spherically symm. $\vec{E}_i$ of charge q_i

As $\Phi_{E_i} = \oint_S \vec{E}\cdot d\vec{a} = (4\pi\varepsilon_0)^{-1}\oint \frac{q_i}{r_i^2}\hat{r}_i\cdot(r^2\sin\theta\, d\theta\, d\phi\,\hat{r}) = q_i/\varepsilon_0$, we have $\Phi_E = \oint \vec{E}\cdot d\vec{a} = \sum_i (q_i/\varepsilon_0) = \frac{1}{\varepsilon_0}Q_{enc}$

If $q_i \longrightarrow dq$ we have:

$$\boxed{\begin{array}{l}\text{Integral form of Gauss Law: } \oint \vec{E}\cdot d\vec{a} = \int_V (\rho/\varepsilon_0)\,d\tau \\ \text{Differential form of Gauss Law: } \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0\end{array}}$$

Divergence of $\vec{E}$:

$\vec{\nabla}\cdot\vec{E} = (4\pi\varepsilon_0)^{-1}\int_V \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\vec{r}')\,d\tau$

$= (4\pi\varepsilon_0)^{-1}\int_V 4\pi\,\delta^3(\vec{r}-\vec{r}')\rho(\vec{r}')\,d\tau$

$= \rho(\vec{r}')/\varepsilon_0$

N.B.1. To fully exploit Gauss Law:

1) Choose surface S of constant value $\vec{E}$

e.g. Sphere for point charge or uniformly charged sphere

Pillbox for uniformly charged sheet

2) Integrate over surface and extract constant $|E|$

3) Solve integrals and rearrange to find $|\vec{E}|$

N.B.2. If multiple objects treat separately and invoke superposition

Curl of Electric Field

In spherical coordinates: $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$

$\vec{E}_i\cdot d\vec{l} = q_i(4\pi\varepsilon_0)^{-1}\frac{1}{r_i^2}dr_i \Longrightarrow \oint \vec{E}_i\cdot d\vec{l} = 0$

Stokes Theorem: $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \oint \vec{E}_i\cdot d\vec{l}$

i.e. $\vec{\nabla}\times\vec{E}_i = 0$ and thus $\vec{\nabla}\times\vec{E} = 0$

Electric Potential

As $\vec{\nabla} \times \vec{E} = 0$ the line integral of $\vec{E}$ is independent of the path

Define Potential w.r.t. reference point $\mathcal{O}$ as follows:

$$V(\vec{r}) = -\int_{\mathcal{O}}^{r} \vec{E} \cdot d\vec{l}$$

$$V(\vec{r}_b) - V(\vec{r}_a) = \int_{r_a}^{r_b} (\vec{\nabla} V) \cdot d\vec{l} = -\int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} \implies \vec{E} = -\vec{\nabla} V$$

N.B.1. $\mathcal{O}$ is generally ∞ such that $V(\infty) = 0$

N.B.2. Potentials obey superposition principle

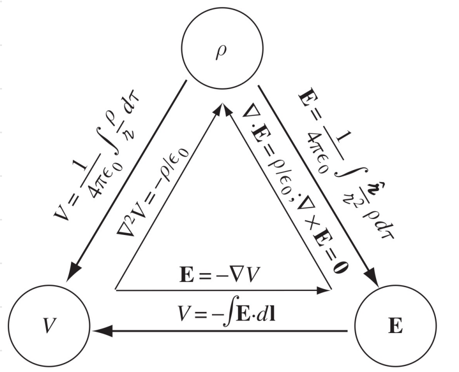
For localised charge distribution:

$$V(\vec{r}) = (4\pi\varepsilon_0)^{-1} \sum_j \frac{q_j}{\mathscr{r}_j} = (4\pi\varepsilon_0)^{-1} \int \frac{dq}{\mathscr{r}}$$

Potentials include both divergence and curl of $\vec{E}$ naturally:

1. $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} V) = -\nabla^2 V$ or $\nabla^2 V = -\rho/\varepsilon_0$
2. $\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V) = 0$

Relationship Summary



Boundary Conditions

Check Book

Work and Energy

Potential difference between $\vec{r}_a$ and $\vec{r}_b$ is the work per unit charge required to carry a particle from point $\vec{r}_a$ to point $\vec{r}_b$ i.e. $W = Q[V(\vec{r}_b) - V(\vec{r}_a)]$

If $\vec{r}_a \longrightarrow \infty$ and $\mathcal{O} = \infty$ we have $W = QV(\vec{r}_b) = QV(\vec{r}) \implies$ Energy required to bring charge into the system from infinity

However, to assemble system one charge at a time we have: $W = (4\pi\varepsilon_0)^{-1} \sum_{i=1}^{n} \sum_{j>1}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$ as adding a new charge affects potential for next charge

↳ Rearrange into $W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j \neq i}^{n} (4\pi\varepsilon_0)^{-1} \frac{q_j}{\mathscr{r}_{ij}} \right) = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i) \longrightarrow$ energy of creation of the config.

↳ The $(1/2)$ is to avoid double counting

↳ $V(\vec{r}_i)$ refers to the total potential (once system is assembled) at location of q_i

For a continuous charge distribution: $W = \frac{1}{2} \int \rho V \, d\tau$ N.B. integration over any volume will work as terms with $\rho = 0$ cancel

↳ Substituting $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$ and integrating by parts we get:

$$W = \frac{1}{2} \varepsilon_0 \left[\int_V E^2 d\tau + \oint_S V \vec{E} \cdot d\vec{a} \right]$$

↳ As any volume works and E^2 term increases with increasing volume:

$$W = \frac{1}{2} \varepsilon_0 \int E^2 d\tau \quad \text{(Over all-space)}$$ i.e. Energy density of field: $\frac{1}{2} \varepsilon_0 E^2$

↳ energy stored by field

$\longrightarrow$ This is the total energy stored in config. i.e. energy of creation of (charges + config)

N.B. Superposition does not work on W (i.e. $W = \sum W_i$) as there are E^2 and $V\vec{E}$ integrals

Conductors

Look at book

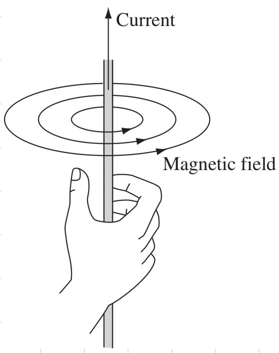
Chapter 5 - Magnetostatics

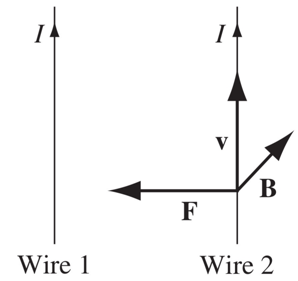
Magnetic Fields

Due magnetic forces, parallel currents attract while anti-parallel currents repel

↳ A stationary charge generates an electric field $\vec{E}$

A moving charge (i.e. current) generates $\vec{E}$ and $\vec{B}$





Magnetic Force

$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$ i.e. Lorentz Force Law

$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) \Longrightarrow$ Axiom of theory, can only be derived from pr. of least action

Magnetic forces do no work i.e. $W_{mag} = 0$

↳ $dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}\, dt = 0$

Currents

Current := Charge per unit time   Unit: $1\,A = 1\,C/s$

Current is a vector oriented along the direction of motion of charges

↳ e.g. line charge λ travelling through wire with speed $\vec{v}$: $\vec{I} = \lambda \vec{v}$

↳ $\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{I} \times \vec{B})\, dl$

Total current crossing surface S: $I = \int_S J\, da_\perp = \int_S \vec{J} \cdot d\vec{a}$

Total current leaving volume V: $I = \oint \vec{J} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{J})\, d\tau$

$I = -\frac{d}{dt} \int_V \rho\, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$

Continuity equation: $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

In general:

$\vec{F}_{mag} = \int I\,(d\vec{l} \times \vec{B})$ Line current $\vec{I} = \lambda \vec{v}$

$\vec{F}_{mag} = \int (\vec{K} \times \vec{B})\, da$ Surface current $\vec{K} = \sigma \vec{v}$

$\vec{F}_{mag} = \int (\vec{J} \times \vec{B})\, d\tau$ Volume current $\vec{J} = \rho \vec{v}$

Biot-Savart Law

Stationary charges	⇒	constant electric fields: electrostatics.
Steady currents	⇒	constant magnetic fields: magnetostatics.

↳ Continuous, constant flow of charge i.e. $\overbrace{(\partial\rho/\partial t) = (\partial\vec{J}/\partial t) = 0}^{(\vec{\nabla}\cdot\vec{J}) = (\partial\vec{J}/\partial t) = 0}$

↳ moving charge is not a steady current

Biot-Savart Law (only for Steady Curr.):

$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dl = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ Line Current $\vec{I} = \lambda \vec{v}$

$\mu_0 = 4\pi \cdot 10^{-7}\ N/A^2$

$1\,T = 1\,N/(A \cdot m)$

$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$ Surface Current $\vec{K} = \sigma \vec{v}$

$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$ Volume Current $\vec{J} = \rho \vec{v}$

Divergence and Curl of $\vec{B}$

Using Biot-Savart Law:

1. $\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau' = 0 \Longrightarrow \vec{\nabla} \cdot \vec{B} = 0$ as divergence of a curl is always zero

2. $\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau' \Longrightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ or equivalently $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ where $I_{enc} = \int_S \vec{J} \cdot d\vec{a}$

Ampere's Law is always valid, just like Gauss Law, but not always useful

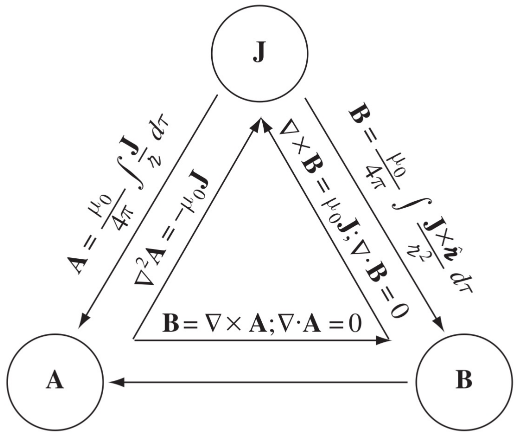
↳ Use gauss law only when symmetry allows to tak $\vec{B}$ out of the integral

(1) infinite straight lines (prototype: Ex. 5.7),
(2) infinite planes (prototype: Ex. 5.8),
(3) infinite solenoids (prototype: Ex. 5.9),
(4) toroids (prototype: Ex. 5.10).

Vector Potential

Just as one can define $\vec{E} = -\vec{\nabla} V$ to have $\vec{\nabla} \times \vec{E} = 0$,

one can define the vector potential $\vec{A}$ to have $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{\nabla} \cdot \vec{B} = 0$



Curl of $\vec{B}$: $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$

We can add a the gradient of a scalar λ with no effect on $\vec{B}$ $(\vec{\nabla} \times \vec{\nabla} \lambda = 0)$

λ can always be chosen such that $\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2 \lambda$

↳ If $\vec{\nabla} \cdot \vec{A}_0$ goes to zero at infinity: $\lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_0}{\mathcal{r}} d\tau'$

↳ For $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ and $\vec{J} \longrightarrow 0$ as $\vec{r}' \longrightarrow \infty$:

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{\mathcal{r}} d\tau'$ N.B. Cannot be used for open steady currents as $\vec{J} =$ con

N.B.2. Ordinarily $\vec{A} \parallel \vec{J}$

Lecture 08.02.2024

Maxwell's Equations

1. $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ — Gauss — (Flux of $\vec{E}$ through closed surface) = (Charge inside)/ε_0
2. $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$ — Faraday — (Line integral of $\vec{E}$ around a loop) = $-\frac{d}{dt}$ (Flux of $\vec{B}$ through loop)
3. $\vec{\nabla}\cdot\vec{B} = 0$ — No mag. monopoles — (Flux of $\vec{B}$ through closed surface) = 0
4. $c^2\vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial\vec{E}}{\partial t}$ — Ampère - Maxwell — c^2 (Integral of $\vec{B}$ around a loop) = (Current through loop)/ε_0 + $\frac{d}{dt}$ (Flux of $\vec{E}$ through the loop)

Conservation of charge: $\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t$ — (Flux of current through a closed surface) = $-\frac{d}{dt}$ (Charge inside)

Force Law: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

→ Disappears in fundamental physics, Hamiltonian is a much stronger statement

However, Lagrangian is even more fundamental as Hamiltonian is non-relativistic and Lagrangian stems from principle of least action

② and ③ are homogeneous

① and ④ are inhomogenous i.e. have charge

While Maxwell's equations are invariant under many symmetry operation they are not manifestly Lorentz invariant

Vectors

A vector is an element of a vector space

e.g. In Vector Space $\mathbb{R}^3$ we have $\vec{x}$, $\vec{J}$, $\vec{E}$, $\vec{B}$

In Vector Space $\mathbb{R}^4$ we have x^μ, J^μ, A^μ

Vectors and Tensors are covariant or contravariant i.e. change in one way or another when a transf.

Scalars are always Lorentz Invariant

However, to write $\vec{E}$ and $\vec{B}$ in covariant form we need tensor $F^{\mu\nu}$ $\xrightarrow{\text{Applying Lorentz Transform Mixes } \vec{E} \text{ and } \vec{B}}$ Maxwell's equations need to be written in Lorentz inv. form

Inhomogeneous Maxwell eq: $\partial_\mu F^{\mu\nu} = J^\nu$ ⟹ Covariant both sides

Homogegenous Maxwell eq: $\partial_\mu(\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}) = 0$ ⟹ Zero vector is invariant

QED

$\vec{E}$ and $\vec{B}$ are operators with quantised eigenvalue

Currents are reduced to charges of elementary particles

⟹ Maxwell only valid for macroscopic situations

Photons are the quanta (no charge, no mass but $s=1$)

Potentials

$\vec{E} = -\vec{\nabla}V = \frac{\partial\vec{A}}{\partial t}$ — Electric Potential V

$\vec{B} = \vec{\nabla}\times\vec{A}$ — Auxiliary Field $\vec{A}$: Thought to not be fundamental

$\vec{E}$ and $\vec{B}$ are emergent

$V, \vec{A}$ are fundamental and form A^μ i.e. the photon field

Photon field: $A^\mu = (V, \vec{A})$

Requirement of Gauge Invariance: $\partial_\mu A^\mu = 0$

⟹ Equations of QED do not have $\vec{E}, \vec{B}$ but rather A^μ

QED = QM + SR and QED is an example of QFT

→ $\hat{x}$ is an operator, t is a parameter

→ x and t are on equal footing

→ As photons have $m=0$ $S=1$ we only have two polarisations

2 degrees of freedom, V and $\vec{A}$ comp. must be coupled

To merge QM and SR, two options:

① Promote t to operator ⟶ $[\hat{E}, \hat{t}] \neq 0$ i.e. $-\infty < E < \infty$ Impossible!

② Demote x to parameter ⟶ Fields become operators e.g. $A^\mu(\vec{x}, t)$ is a quantum Lorentz inv. field

Lecture 12.02.2024

Review Chapter 7 and 10

Maxwell's Equations

1. Gauss: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ — In words: Flux of $\vec{E}$ through surface is proportional to enclosed charge
2. Ampère: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ — In words: Rotation of $\vec{B}$ is proportional to current + displacement current
3. Faraday: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ — In words: Rotation of $\vec{E}$ is equal to the rate of change in time of $\vec{B}$
4. Anti-Dirac: $\vec{\nabla}\cdot\vec{B} = 0$ — In words: Flux of $\vec{B}$ through surface is always zero $\Longrightarrow$ No monopole sources of $\vec{B}$

+ Continuity Equation: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ — In words: Change over time in enclosed charge must correspond to an (in/out) flowing current
$\hookrightarrow$ Total charge is conserved

The displacement current ensures charge conservation i.e. if it were missing $\vec{\nabla}\cdot\vec{J} = 0$ and by continuity equation $\partial_t\rho = 0$
Thus, Ampère-Maxwell $\Longleftrightarrow$ Continuity equation

Derivations Related to continuity equation

① Conserved Charge i.e. $dQ/dt = 0$

By Continuity equation: $\partial\rho/\partial t = -\vec{\nabla}\cdot\vec{J}$

Charge enclosed by volume V with surface S: $Q = \int_V d^3x\,\rho$

As Q is integrated over space there is no explicit x^i dependence: $\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x^i}\frac{\partial x^i}{\partial t} = \frac{\partial Q}{\partial t}$

Then: $\frac{dQ}{dt} = \int_V d^3x\,\frac{\partial\rho}{\partial t} = -\int_V d^3x\,\vec{\nabla}\cdot\vec{J}$

By the divergence Theorem: $\frac{dQ}{dt} = -\int_S \vec{J}\cdot d\vec{a}$

$\hookrightarrow$ The change over time of the charge enclosed by volume is equal to the current escaping through the surface

If $V \longrightarrow$ all space, $S \longrightarrow \infty$ and there is no current escaping the through the surface i.e. $\vec{J} = 0$ at ∞ $\Longrightarrow$ Q is conserved

② Maxwell's Displacement current:

Gauss Law: $\vec{\nabla}\cdot\vec{E} = -\rho/\varepsilon_0$

No magnetic Monopoles: $\vec{\nabla}\cdot\vec{B} = 0$

Continuity Equation: $\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$

Then: $-\varepsilon_0\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = -\vec{\nabla}\cdot\vec{J}$ or $\vec{\nabla}\cdot\left[\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right] = 0$

As the divergence of a Curl is always zero:

$\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times(\vec{B}/\mu_0)$ Identified by units and Ampère's Law

Maxwell's Equations in Potential Formulation

The curl of a gradient is always zero i.e. $\vec{\nabla}\times(\vec{\nabla}V) = 0$

The divergence of curl is always zero i.e. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

Curl of Curl: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

Then: $\vec{B} = \vec{\nabla}\times\vec{A}$ (such that $\vec{\nabla}\cdot\vec{B} = 0$) and $\vec{E} = -\vec{\nabla}V - \left(\frac{\partial\vec{A}}{\partial t}\right)$

Gauss Law becomes: $\vec{\nabla}\cdot(\vec{\nabla}V) + \vec{\nabla}\cdot\frac{\partial\vec{A}}{\partial t} = \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0$

Ampère's Law becomes: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right)$

Thus Maxwell's equations can be written as:

1. Gauss: $\nabla^2 V + (\partial/\partial t)(\vec{\nabla}\cdot\vec{A}) = -\varepsilon_0^{-1}\rho$
2. Ampère: $\nabla^2\vec{A} - \mu_0\varepsilon_0(\partial^2\vec{A}/\partial t^2) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0(\partial V/\partial t)) - \mu_0\vec{J}$
3. Faraday: $\vec{E} = -\vec{\nabla}V - (\partial\vec{A}/\partial t)$
4. Anti-Dirac: $\vec{B} = \vec{\nabla}\times\vec{A}$

Thankfully this theory presents Gauge Freedom i.e. We can set $(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0(\partial V/\partial t)) = 0$ ("Lorenz Gauge") without consequences

Thus Maxwell's equations can be written as:

1. Gauss: $\Box^2 V = \rho/\epsilon_0$
2. Ampère: $\Box^2 \vec{A} = \mu_0 \vec{J}$
3. Faraday: $\vec{E} = -\vec{\nabla} V - (\partial \vec{A}/\partial t)$
4. Anti-Dirac: $\vec{B} = \vec{\nabla} \times \vec{A}$ $\qquad \partial_\mu \partial^\mu A^\nu$

N.B. 1. and 2. are now "Inhomogeneous Wave Equations":

- The terms ρ/ϵ_0 and $\mu_0 \vec{J}$ are called source terms
- The $\Box^2 = -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ operator is known as d'Alambertian

Remains valid in all inertial reference frames

The $\Box^2$ operator can be written as $\Box^2 = \partial_\mu \partial^\mu$ where $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$. Therefore, $\Box^2$ is Lorentz Invariant

By exploiting $A^\mu = (V, \vec{A})$ and $J^\mu = (\rho, \vec{J})$ we can combine 1 and 2 into one single equation: $\partial_\nu \partial^\nu A^\nu = \mu_0 J^\nu$

As both A^μ and J^μ transform as vectors under Lorentz transformations and $\partial_\mu \partial^\mu$ is Lorentz invariant, this formula is manifestly Lorentz Covariant

Gauge Transformation and Freedom

$\vec{A}$ and V are completely determined by equation (3) and (4) of the Maxwell's set. As such, we are free to choose any $\vec{A}$ and V as long as it does not affect $\vec{E}$ and $\vec{B}$

The application of transformations on $\vec{A}$ and V that do not alter $\vec{E}$ and $\vec{B}$ are called "Gauge Transformations"

Which transformations are Gauge transformations?

$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\alpha}$, $\quad \vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'$

$V \longrightarrow V' = V + \beta$, $\quad \vec{E} = -\vec{\nabla} V - (\partial \vec{A}/\partial t) = -\vec{\nabla} V' - (\partial \vec{A}'/\partial t)$

As a result: $\vec{\nabla} \times \vec{\alpha} = 0$ and $\vec{\nabla} \beta + (\partial \vec{\alpha}/\partial t) = 0$

We can thus write $\vec{\alpha} = \vec{\nabla} f$ where f is any scalar function

It follows that $\vec{\nabla}(\beta + \partial f/\partial t) = \vec{\nabla} k(t) = 0$ where $k(t)$ is any function independent of position

A Gauge transformation is thus any transformation that adds $\vec{\nabla} f$ to $\vec{A}$ and adds $k(t) - \frac{\partial f}{\partial t}$ to V

Coulomb Gauge

Set $\vec{\nabla} \cdot \vec{A} = 0$

Maxwell equations become:

1. Gauss: $\nabla^2 V = -\rho/\epsilon_0$
2. Ampère: $\nabla^2 \vec{A} - \mu_0 \epsilon_0 (\partial^2 \vec{A}/\partial t^2) = \mu_0 \epsilon_0 \vec{\nabla}(\partial V/\partial t) - \mu_0 \vec{J}$

$\Longrightarrow$ Poisson's equation: $V(\vec{x}, t) = \int \frac{\rho(\vec{x}', t)}{R} d^3x'$

The potential changes instantaneously as charge configuration changes, which is not possible in SR. This is due to the lack of Lorentz inv. in Coulomb gauge. Retardation is still maintained by $\vec{A}$ and thus $\vec{E}$ and $\vec{B}$ present retardation

3. Faraday: $\vec{E} = -\vec{\nabla} V - (\partial \vec{A}/\partial t)$
4. Anti-Dirac: $\vec{B} = \vec{\nabla} \times \vec{A}$

As a result, Maxwell equations are not manifestly Lorentz invariant with the Coulomb Gauge

Nonetheless, one can recover the previous equations by splitting $\vec{J}$ into:

- Longitudinal / Irrotational Current: $\vec{J}_\ell$ with $\vec{\nabla} \times \vec{J}_\ell = 0$
- Transversal / Solenoidal Current: $\vec{J}_t$ with $\vec{\nabla} \cdot \vec{J}_t = 0$

As such $\vec{J}_\ell$ can be written as $\vec{J}_\ell = -\vec{\nabla} f$, $[f] = C/(m s)$ such that $\vec{\nabla} \times \vec{J}_\ell = 0 \ \forall f$

As we have the freedom of choosing any f has long as the units are correct, we choose $f = \epsilon_0 \frac{\partial V}{\partial t}$ where $[V] = N \cdot m/C$ and $[\epsilon_0] = C^2/(N m^2)$ such that $[f] = C/ms$

As a result:

$\vec{J} = \vec{J}_\ell + \vec{J}_t \Longrightarrow$ Ampère: $\nabla^2 \vec{A} - \mu_0 \epsilon_0 (\partial^2 A^2/\partial t^2) = -\mu_0 \vec{J}_t$

As a result, the photon only presents the 2 trans l degrees of freedom/polarization and no longitudinal polarizations as it only couples to $\vec{J}_t$

Defining the Wave Equation

General wave-equation: $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{x},t)$ where ψ is the wave-function and $f(\vec{x},t)$ are source terms

We can go from time to angular frequency ω and viceversa by applying a fourier transform:

$$\psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{x},\omega)e^{-i\omega t}d\omega \qquad f(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vec{x},\omega)e^{-i\omega t}d\omega$$

$$\psi(\vec{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{x},t)e^{i\omega t}dt \qquad f(\vec{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vec{x},t)e^{i\omega t}dt$$

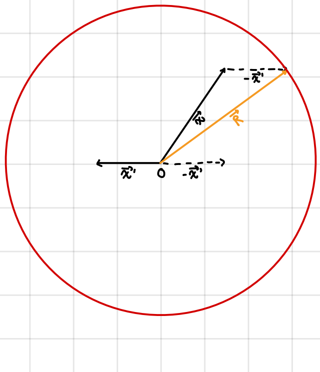
By plugging this into the wave equation: $(\nabla^2+k^2)\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega)$ "Helmholtz Equation" with $k=\omega/c$

Let's now introduce the Green function $G_k(\vec{x},\vec{x}')$ where $\vec{x}$ is the position that we are interested in and $\vec{x}'$ is the position of the source object e.g. charge

The Green Function is the impulse response of an inhomogeneous linear differential operator L i.e. G satisfies $LG=\delta$

Since the source distribution is a sum of δ-functions one can solve $LG=\delta$ first and then exploit the superposition principle to determine the complete solution.

In this case: $(\nabla^2+k^2)G_k(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$ or $(\nabla^2+k^2)G_k(\vec{R}) = -4\pi\delta(\vec{R})$ where $\vec{R}=\vec{x}-\vec{x}'$



New coordinate system in $\mathbb{R}^2$

As G_k is a function of $\vec{R}$, we can switch to spherical coordinates: $\nabla^2 f = \frac{1}{R^2}\left(\frac{\partial^2}{\partial R^2}\right)(Rf)+\dots$

Then, Helmholtz equation becomes: $\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2G_k = -4\pi\delta(\vec{R})$

$\hookrightarrow$ This equations turns into a homogeneous one when $\vec{R}\neq 0$

$(d^2/dR^2)(RG_k) + k^2(RG_k) = 0$

The general solution becomes: $RG_k = Ae^{ikR} + Be^{-ikR}$

In general: $G_k(\vec{R}) = AG^{+}(\vec{R}) + BG^{(-)}(\vec{R})$ where $G^{(\pm)}(R) = e^{\pm ikR}/R$

If $\vec{R} \ll 1$, $\lim_{kR\to 0} RG_k = (A+B) \approx 1$ and $G_k \longrightarrow 1/R$

To include time dependence we replace $f(\vec{x},t)$ with $\delta(\vec{x}-\vec{x}')\delta(t-t')$ such that $f(\vec{x},\omega) = \delta^3(\vec{x}-\vec{x}')e^{i\omega t}$

Then, the general equation becomes: $(\nabla^2+k^2)G_k(\vec{R},\omega) = -4\pi\delta(\vec{R})e^{i\omega t'}$

$\hookrightarrow$ If $\vec{R}\neq 0$, the solution becomes: $G_k(\vec{R};\omega) = AG^{(+)}(R,\omega) + BG^{(-)}(\vec{R},\omega)$, $G_k^{(\pm)}(R,\omega) = G_k^{(\pm)}(R)e^{i\omega t'}$ where

Doing a Fourier transform: $G_k^{(\pm)}(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ikR}}{R}e^{-i\omega\tau}d\omega$ with $\tau = t-t'$ $\Longrightarrow$ $G_k^{(\pm)}(\vec{R},\tau) = \frac{1}{R}\delta(\tau\mp R/c)$

Then we have two solutions: $G^{(\pm)}(\vec{x},t;\vec{x}',t') = \dfrac{\delta\left[t'-\left(t\mp\frac{|\vec{x}-\vec{x}'|}{c}\right)\right]}{|\vec{x}-\vec{x}'|}$, retarded time: $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$

- G^{+} is the advanced solution
- G^{-} is the retarded solution

Then:

$$V(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int\frac{d^3\vec{x}'[\rho(\vec{x}',t')]_{ret}}{R}$$

$$A(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3\vec{x}\,[\vec{J}(\vec{x}',t')]_{ret}R^{-1}$$

ret ≡ Evaluated at retarded time

General solution to wave-equation by means of Green's Function:

$$\psi^{(\pm)}(\vec{x},t) = \int G^{(\pm)}(\vec{x},t;\vec{x}',t')f(\vec{x}',t')\,d^3x'dt'$$

$\hookrightarrow$ $\psi(\vec{x},t) = \int\frac{[f(\vec{x}',t')]_{ret}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'$

Lecture 15.02.2023

Retarded Potentials

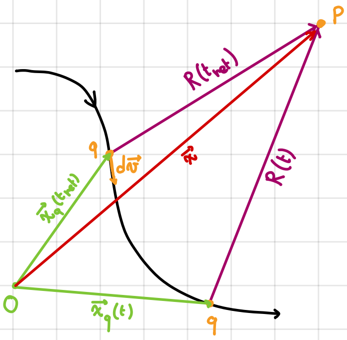
As electrostatic waves travel at the speed of light, changes in charge and current density do not reflect instantaneously on fields and potentials
i.e. Information travels at finite velocity and thus changes in potential / fields reflect configurations at earlier times

The time it takes for information to travel a distance R is given by $\Delta t = R/c$
The earlier time that needs to be considered is the "retarded time" $t_{ret} = t - \Delta t = t - R/c$

$\vec{x} \equiv$ Position at which potential is evaluated
$\vec{x}' \equiv$ Source position
$\vec{R} = \vec{x} - \vec{x}'$
ret $\equiv$ Evaluated at retarded time i.e. $t' = t_{ret}$
The general solution to Maxwell's Potential equations:

$$V(\vec{x}, t) = (4\pi\varepsilon_0)^{-1} \int d^3x' \frac{[\rho(\vec{x}', t')]_{ret}}{R}$$

$$\vec{A}(\vec{x}, t) = (\mu_0/4\pi) \int d^3x' \frac{[\vec{J}(\vec{x}', t')]_{ret}}{R}$$

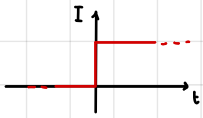


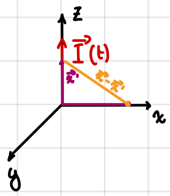
In this case:
$$\rho(\vec{x}', t') = q\,\delta(\vec{x}' - \vec{x}_q(t'))$$
$$\vec{J}(\vec{x}', t') = q\,\vec{v}(t')\,\delta(\vec{x}' - \vec{x}_q(t'))$$

N.B. Advanced solutions (i.e. $t_{ad} = t + R/c$) are equally solutions to Maxwell's equations
However they are unphysical as they violate the principle of causality

Example 10.2

Question: Infinite straight wire carries current $I(t) = I_0\,\Theta(t)$
Find $\vec{E}$ and $\vec{B}$





Solution: As $\rho(\vec{x}', t') = 0 \;\; \forall \vec{x}', t'$ we are left with $\vec{A}(\vec{x}', t')$
Current density given by $\vec{J}(\vec{x}', t') = I(t)\,\delta(x)\,\delta(y)$
Wire has cylindrical symmetry such that $R^2 = |\vec{x} - \vec{x}'|^2 = s^2 + z^2$

Integration boundaries:
$t_{ret} = t - R/c \implies z = \pm\sqrt{c(t - t_{ret})^2 - s^2}$
As only from $t_{ret} > 0$ we have a non-zero $I(t)$ we get: $z = \pm\sqrt{ct - s^2}$

Then: $\vec{A}(s,t) = \frac{\mu_0}{4\pi} \int \frac{[\vec{J}(\vec{x}', t')]_{ret}}{\sqrt{s^2 + z^2}} d^3x' = \frac{\mu_0 I_0}{2\pi} \int_0^{\sqrt{ct - s^2}} \frac{\hat{z}\, dz}{\sqrt{s^2 + z^2}}$
As $2\cosh(u) = e^u + e^{-u}$ and $2\sinh(u) = e^u + e^{-u}$ we have $\cosh^2(u) - \sinh^2(u) = 1$
$\longrightarrow$ $z = s\sinh(u) \implies dz = s\cosh(u)\,du$ and $2(z/s)e^u - e^{2u} + 1 = 0$
Then: $e^u = (z/s) \pm \sqrt{(z/s)^2 + 1}$ or $u = \ln\left((z/s) \pm \sqrt{(z/s)^2 + 1}\right)$
As $u \geq 0$, $u = \ln\left((z/s) + \sqrt{(z/s)^2 + 1}\right)$
Also: $dz / \sqrt{s^2 + z^2} = du$

Then: $\vec{A}(s,t) = \frac{\mu_0 I_0}{2\pi} \ln\left((z/s) + \sqrt{(z/s)^2 + 1}\right)\Big|_0^{\sqrt{(ct)^2 - s^2}} = \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct}{s} + \sqrt{\left(\frac{ct}{s}\right)^2 - 1}\right)\hat{z}$
The field are:

$$\mathbf{E}(s,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2 - s^2}}\hat{\mathbf{z}},$$

As $t \longrightarrow \infty$, $\vec{E} \longrightarrow 0$

$$\mathbf{B}(s,t) = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2 - s^2}}\hat{\phi}.$$

As $t \longrightarrow \infty$, $\vec{B} \longrightarrow \frac{\mu_0 I}{2\pi s}\hat{\phi}$

Jefimenko's Equations

Exploiting $\vec{E} = -\vec{\nabla}V - (\partial\vec{A}/\partial t)$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ one can write maxwell's equations as follows:

1. $\Box^2\vec{E} = -\left(\vec{\nabla}\rho + \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}\right)/\varepsilon_0$
2. $\Box^2\vec{B} = \mu_0\vec{\nabla}\times\vec{J}$

Proof:

$$\nabla^2\vec{E} = -\nabla^2\left[\vec{\nabla}V + (\partial\vec{A}/\partial t)\right] = -\vec{\nabla}(\nabla^2 V) - \nabla^2(\partial A/\partial t)$$

$$(\partial^2\vec{E}/\partial t^2) = -\vec{\nabla}(\partial^2 V/\partial t^2) - (\partial^3\vec{A}/\partial t^3)$$

$$\begin{aligned}\nabla^2\vec{E} - \mu_0\varepsilon_0(\partial^2\vec{E}/\partial t^2) &= -\vec{\nabla}\left[\nabla^2 V - \mu_0\varepsilon_0(\partial^2 V/\partial t^2)\right] - \frac{\partial}{\partial t}\left[\nabla^2\vec{A} - \mu_0\varepsilon_0(\partial^2\vec{A}/\partial t^2)\right] = \\ &= \vec{\nabla}(\Box^2 V) + (\partial/\partial t)(\Box^2\vec{A}) = \\ &= (\vec{\nabla}\rho/\varepsilon_0) + \mu_0(\partial\vec{J}/\partial t) = \\ &= \left[\vec{\nabla}\rho + c^{-2}(\partial\vec{J}/\partial t)\right]/\varepsilon_0\end{aligned}$$

As these equations are inhomogeneous wave equations the solution can be found by using the general formula (page 4)
The solutions are known as Jefimenko's equations
The Jefimenko's equations are:

From wave equation: $\vec{E} = -\frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{R}\left[\vec{\nabla}'\rho + \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t'}\right]_{ret}$ $\qquad$ $\vec{B} = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\vec{\nabla}'\times\vec{J}\right]_{ret}$

From potentials:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}\left[\rho(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{cR^2}\left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\right\} \qquad \vec{B} = -\frac{\mu_0}{4\pi}\int d^3x'\left\{\frac{\hat{R}}{R^2}\times\left[\vec{J}(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{cR}\times\left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\right\}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}\left[\rho(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{cR}\left[\dot{\rho}(\vec{x}',t')\right]_{ret} - \frac{1}{c^2R}\left[\dot{\vec{J}}(\vec{x}',t')\right]_{ret}\right\} \qquad \vec{B} = -\frac{\mu_0}{4\pi}\int d^3x'\left\{\frac{\hat{R}}{R^2}\times\left[\vec{J}(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{cR}\times\left[\dot{\vec{J}}(\vec{x}',t')\right]_{ret}\right\}$$

Note: $\vec{\nabla}'[f]_{ret} \neq [\vec{\nabla}'f]_{ret}$

Derivation of Jefimenko's equations:

The retarded potentials are: $\qquad$ $\vec{R} = \vec{x} - \vec{x}'$ such that $x'_j = x_j - R_j$

$V(\vec{x},t) = (4\pi\varepsilon_0)^{-1}\int d^3x'\,\frac{\rho(\vec{x}',t_r)}{R}$ $\qquad$ $t_r = t - R/c$

$\vec{A}(\vec{x},t) = (\mu_0/4\pi)\int d^3x'\,\frac{\vec{J}(\vec{x}',t_r)}{R}$

The gradient of $V(\vec{x},t)$ is: $\vec{\nabla}V = (4\pi\varepsilon_0)^{-1}\int d^3x'\,\vec{\nabla}\left[\frac{\rho(\vec{x}',t_{ret})}{R}\right] = (4\pi\varepsilon_0)^{-1}\int d^3x'\left[(\vec{\nabla}\rho(\vec{x}',t_r))R^{-1} + \rho(\vec{x}',t_r)(\vec{\nabla}R^{-1})\right]$

The $(\partial\vec{A}/\partial t)$ is: $(\partial\vec{A}/\partial t) = (\mu_0/4\pi)\int d^3x'\left(\frac{\partial}{\partial t}\right)\left[\frac{\vec{J}(\vec{x}',t_r)}{R}\right] = (\mu_0/4\pi)\int d^3x'\left[\frac{\partial\vec{J}(\vec{x}',t_r)}{\partial t}R^{-1} + \vec{J}(\vec{x}',t_r)\underbrace{\frac{\partial R^{-1}}{\partial t}}_{0}\right] = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t}\right]_{ret}$ as $(\partial t_r/\partial t) = 1$

The curl of $\vec{A}$ is: $\vec{\nabla}\times\vec{A} = (\mu_0/4\pi)\int d^3x'\,\vec{\nabla}\times\left[\frac{\vec{J}(\vec{x}',t_r)}{R}\right] = (\mu_0/4\pi)\int d^3x'\left[\frac{\vec{\nabla}\times\vec{J}(\vec{x}',t_r)}{R} + (\vec{\nabla}R^{-1})\times\vec{J}(\vec{x}',t_r)\right]$

Computing R related quantities:

As $\vec{R}$ is in radial direction $\hat{R}$, we can use spherical coordinates to have: $\vec{\nabla}R^{-1} = -R^{-2}\hat{R}$ and $\vec{\nabla}\times\vec{R} = 0$

In addition, $R = c(t-t_r)$ so $\partial R^{-1}/\partial t = -R^{-2}(\partial R/\partial t) = -cR^{-2}$

Charge density gradient:

$$\vec{\nabla}\rho(\vec{x}',t_r) = \frac{\partial\rho(\vec{x}',t_r)}{\partial x'_i}\overbrace{\frac{\partial x'_i}{\partial x_j}}^{0}\hat{x}_j + \frac{\partial\rho(\vec{x}',t_r)}{\partial t_r}\frac{\partial t_r}{\partial x_j}\hat{x}_j = \dot{\rho}(\vec{x}',t_r)\,\vec{\nabla}t_r = \dot{\rho}(\vec{x}',t_r)\,\vec{\nabla}(t-R/c) = -\frac{1}{c}\dot{\rho}(\vec{x}',t_r)\hat{R}$$

Current density curl:

$$\vec{\nabla}\times\vec{J}(\vec{x}',t_r) = \hat{x}^i\varepsilon_{ijk}\partial_j J^k(\vec{x}',t_r) = \hat{x}^i\varepsilon_{ijk}\left[\partial'_\alpha J^k(\vec{x}',t_r)\overbrace{(\partial_j(x')^\alpha)}^{0} + \partial_{t_r}J^k(\vec{x}',t_r)(\partial_j t_r)\right] = \hat{x}^i\varepsilon_{ijk}\dot{J}^k(\vec{x}',t_r)(\partial_j t_r) = -\dot{\vec{J}}(\vec{x}',t_r)\times(\vec{\nabla}t_r) = +(1/c)\dot{\vec{J}}(\vec{x}',t_r)\times\hat{R}$$

It thus follows that:

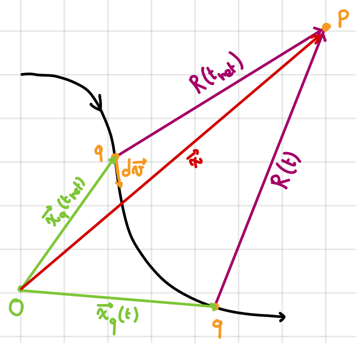
$\vec{\nabla}V = -(4\pi\varepsilon_0)^{-1}\int d^3x'\left\{\frac{\hat{R}}{cR}\left[\dot{\rho}(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{R^2}\left[\rho(\vec{x}',t')\right]_{ret}\right\}$ $\qquad$ $(\partial\vec{A}/\partial t) = (4\pi\varepsilon_0)^{-1}\int d^3x'\,\frac{1}{c^2R}\left[\dot{\vec{J}}(\vec{x}',t')\right]_{ret}$

$\vec{\nabla}\times\vec{A} = (\mu_0/4\pi)\int d^3x'\left\{\frac{\hat{R}}{R^2}\times\left[\vec{J}(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{cR}\times\left[\dot{\vec{J}}(\vec{x}',t')\right]_{ret}\right\}$

The fields are thus:

$$\vec{E}(\vec{x},t) = (4\pi\varepsilon_0)^{-1}\int d^3x' \left\{ \frac{\hat{R}}{R^2}[\rho(\vec{x}',t')]_{ret} + \frac{\hat{R}}{cR^2}[\dot{\rho}(\vec{x}',t')]_{ret} - \frac{1}{c^2R}[\dot{\vec{J}}(\vec{x}',t')]_{ret} \right\}$$

$$\vec{B}(\vec{x},t) = (\mu_0/4\pi)\int d^3x' \left\{ \frac{\hat{R}}{R^2}\times[\vec{J}(\vec{x}',t')]_{ret} + \frac{\hat{R}}{cR}\times[\dot{\vec{J}}(\vec{x}',t')]_{ret} \right\}$$



Jefimenko's Equations for a moving point charge

$$\rho(\vec{x}',t') = q\,\delta(\vec{x}'-x_q(t')) \longmapsto \vec{J}(\vec{x}',t') = \rho(\vec{x}',t')\vec{v}(t')$$

$$\vec{E}(\vec{x},t) = q(4\pi\varepsilon_0)^{-1}\Big\{ \overbrace{[\hat{R}/R^2]_{ret}}^{\mathrm{I}} + \overbrace{([R]_{ret}/c)\partial_t[\hat{R}/R^2]_{ret}}^{\mathrm{II}} + \overbrace{c^{-2}\partial_t^2[\hat{R}]_{ret}}^{\mathrm{III}} \Big\} \qquad \vec{B}(\vec{x},t) = ([\hat{R}]_{ret}\,c^{-1})\times\vec{E}(\vec{x},t)$$

I: This term is the retarded Coulomb field i.e. the Coulomb field evaluated at a retarded time

II: This term is a linear extrapolation of the retarded coulomb field up to the current time t. Its structure is given by $(t-t_r)^*$ · "rate of change of retarded coulomb field"

If field changes slowly enough: I + II ~ instantaneous coulomb field

III: This term is non-vanishing at $r \longmapsto \infty$, results in e.m. propagation

$^*(t-t_r)$ is the retardation delay

Lecture 19.02.2024

Relativity

Special relativity is the manifestation of space time symmetry

- Einstein postulated this symmetry as:
  1) Principle of relativity: All physics laws apply in all inertial frames
  2) Universal Speed of Light: Light has speed c in all frames
- Group Theory: Spacetime is invariant under Poincarè Group

  Poincarè Group (10 Params Lie Group)
  - ↳ 3+1 Translation (Space + Time)
  - ↳ Lorentz Group $SO(1,3)$ (6 Params Lie Group)
    - ↳ 3 Rotations
    - ↳ 3 Boosts

Rotation:

$$\Lambda = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right)$$

$R \in SO(3)$ s.t. $R = R^*$, $R^T = R^{-1}$

$\Lambda = \Lambda^*$, $\Lambda^{-1} = g\Lambda g \Longrightarrow \Lambda$ is pseudo-orthogonal

Boost (in $\hat{x}$)

$$\Lambda = \left(\begin{array}{cc|c} \cosh\eta & -\sinh\eta & \\ \sinh\eta & \cosh\eta & \emptyset \\ \hline \multicolumn{2}{c|}{\emptyset} & I \end{array}\right)$$

$\eta \equiv$ rapidity

Aspects of relativity

Notation:

- $v \equiv$ Velocity in frame S
- $v' \equiv$ Velocity in frame S'
- $v_{SS'} \equiv$ Velocity of S wrt velocity of S'

⟶ Einstein's Velocity addition: $v' = \dfrac{v + v_{SS'}}{1 + (v\, v_{SS'}/c^2)}$

- $\Delta t \equiv$ Time interval in S
- $\Delta t' \equiv$ Time interval in S'

$\beta \equiv v_{SS'}/c$

$\gamma \equiv (1-\beta^2)^{-1/2}$

Simultaneity: Two events that are simultaneous in one frame are generally not simultaneous in another

↳ Observation: Artificial reconstruction of event by observer independent of their location

Time dilation and length contraction

- Time dilation: Moving clocks run slow $\Longrightarrow \Delta t' = \gamma\, \Delta t$
- Length contracted: Moving objects are shortened in direction parallel to velocity $\Longrightarrow \Delta x' = \gamma^{-1} \Delta x$

An observer in a frame S might observe that clocks moving with velocity $v_{SS'}$ run slow and objects are shorted. On the other hand, from the point of view of S' the objects at rest in S are the ones that are shortened and the clocks run slow. Is this a contradiction? No, because the two observer are measuring two different things as measuring systems are also affected by time dilation and length contraction

Vector Notation

Spacetime follows the Minkoski metric $g = \mathrm{diag}(-1,1,1,1)$ and is thus not euclidean

General Tensor: $T^{\mu\nu\ldots}_{\alpha\beta\ldots} \Longrightarrow (T')^{\mu\nu\ldots}_{\alpha\beta\ldots} = \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda}\ldots\Lambda^{\gamma}{}_{\alpha}\Lambda^{\delta}{}_{\beta}\ldots T^{\kappa\lambda\ldots}_{\gamma\delta\ldots}$

Quantities in 4D Spacetime:

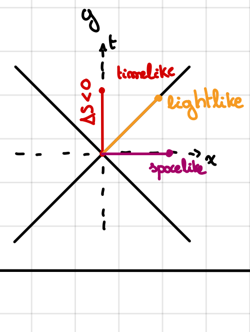
- Rank-0 Tensors: Scalars e.g. $x_\mu x^\mu$, $F_{\mu\nu}F^{\mu\nu}$, ...
  - ↳ Invariant under Lorentz Transformations
- Rank-1 Tensors: Vectors
  - ↳ Contravariant e.g. x^μ, A^μ, ... $\Longrightarrow (A')^\mu = (\partial x'^\mu/\partial x^\nu)A^\nu = \Lambda^\mu{}_\nu A^\nu$
  - ↳ Covariant e.g. x_μ, A_μ, ... $\Longrightarrow (B')_\mu = (\partial x^\nu/\partial x'_\mu)B_\nu = \Lambda_\mu{}^\nu B_\nu$
- Rank-2 Tensors:
  - ↳ Contravariant e.g. $F^{\mu\nu}$ $\Longrightarrow (F')^{\mu\nu} = (\partial x'^\mu/\partial x^\alpha)(\partial x'^\nu/\partial x^\beta)F^{\alpha\beta}$
  - ↳ Covariant e.g. $F_{\mu\nu}$ $\Longrightarrow (F')_{\mu\nu} = (\partial x'^\alpha/\partial x_\mu)(\partial x'^\beta/\partial x_\nu)F_{\alpha\beta}$
  - ↳ Mixed e.g. $F^\mu{}_\nu$, $F_\mu{}^\nu$ $\Longrightarrow (F')^\mu{}_\nu = (\partial x'^\mu/\partial x^\alpha)(\partial x'^\beta/\partial x_\nu)F^\alpha{}_\beta$

N.B. All decent physical quantities should be covariant (i.e. transforms with Lorentz Transform) or be invariant (e.g. scalars)

An example of invariant is the space-time interval

$x^\mu x_\mu = x^2 + y^2 + z^2 - (ct)^2$

$x^\mu = (x^0, \vec{x})^T$ are coordinates of an event in Minkowski spacetime



Derivatives and Identities

Metric Tensor: $g^{\mu\nu} = g_{\mu\nu}$ and $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$

4D - Gradient: $\partial_\mu = (\partial/\partial x^\mu) = (\partial/\partial x^0, \vec{\nabla})$ $\quad \partial^\mu = (-\partial/\partial x^0, \vec{\nabla})$

4 - Divergence: $\partial_\mu A^\mu = \partial^\mu A_\mu = (\partial A^0/\partial x^0) + \vec{\nabla}\cdot\vec{A}$

Field Transformations

Let's consider an electric field $\vec{E}$ and a magnetic field $\vec{B}$ in 3-Dimensions

As we boost from a frame to another we have the following transformations:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \quad \text{and} \quad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp})$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \quad \text{and} \quad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - (\vec{v}/c^2) \times \vec{E}_{\perp})$$

e.g. For a boost from a frame S to a from S' with $\vec{v}_{S'} = v\hat{x}$ we have:

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + (v/c^2)E_z) \qquad B'_z = \gamma(B_z - (v/c^2)E_y)$$

Field Tensor

Field Tensor is $F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$ Dual Field Tensor: $F^{\mu\nu} \longmapsto {}^*F^{\mu\nu}$ if $\vec{E}/c \longmapsto \vec{B}$, $\vec{B} \longrightarrow -\vec{E}/c$

Why is the field tensor like this?

For a boost in x-direction, the 6 entries of $\vec{E}$ and $\vec{B}$ get mixed into 6 new values of $\vec{E}'$, $\vec{B}'$

This cannot be simply expressed by a single vector, however it can be represented by a tensor

This tensor $F^{\mu\nu}$ must be antisymmetric as to have six unique entries

$(F')^{\mu\nu} = \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} F^{\alpha\beta}$ where $\Lambda^{\mu}_{\ \alpha} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for a boost in the x-direction

Then:

$$(F')^{00} = \Lambda^0_{\ 0}\Lambda^0_{\ 0}F^{00} + \Lambda^0_{\ 1}\Lambda^0_{\ 0}F^{10} + \Lambda^0_{\ 0}\Lambda^0_{\ 1}F^{01} + \Lambda^0_{\ 1}\Lambda^0_{\ 1}F^{11}$$

$$(F')^{01} = \Lambda^0_{\ 0}\Lambda^1_{\ 0}F^{00} + \Lambda^0_{\ 1}\Lambda^1_{\ 0}F^{10} + \Lambda^0_{\ 0}\Lambda^1_{\ 1}F^{01} + \Lambda^0_{\ 1}\Lambda^1_{\ 1}F^{11}$$

...

By comparing with eq. for the transformations of the fields we get the Field Tensor and Dual Field Tensor entries

Continuity equation and Maxwell Equations in Tensor Notation

How do ρ and $\vec{J}$ transform?

Consider a frame S' moving with $\vec{v} = u\hat{x}$ wrt S

In S: ρ and $\vec{J} = \rho\vec{u}$ where $\vec{u} = u\hat{x}$ is velocity of charges in S

$\rho = Q/V \propto L^{-3}$ and $\vec{J} \propto L^{-3}\vec{u}$

In S': $V \longmapsto \sqrt{1-(u/c)^2}\,V$ and $u \longmapsto 0$

$\rho \longmapsto \gamma\rho \qquad J_x \longmapsto 0$ and $J_y \longmapsto J_y$, $\vec{J}_z \longmapsto J_z$

As $\Lambda = \left[\begin{array}{cc|c} \gamma & \gamma\beta & \emptyset \\ -\gamma\beta & \gamma & \\ \hline \multicolumn{2}{c|}{\emptyset} & I \end{array}\right]$ and $(J')^{\mu} = \Lambda^{\mu}_{\ \nu} J^{\nu}$:

This results in:

$$(J')^0 = \Lambda^0_{\ 0}J^0 + \Lambda^0_{\ 1}J^1 = \gamma(J^0 + (u/c)J^1)$$

$$(J')^1 = \Lambda^1_{\ 0}J^0 + \Lambda^1_{\ 1}J^1 = \gamma(J^1 - (u/c)J^0)$$

$$(J')^2 = \Lambda^2_{\ 2}J^2 = J^2$$

$$(J')^3 = \Lambda^3_{\ 3}J^3 = J^3$$

If $J^0 = c\rho$ and $J^i = \vec{J}$ we have:

$(J')^0 = c\gamma\rho$, $(J')^1 = 0 \qquad (J')^2 = J_y$ and $(J')^3 = J_z$

The 4-Current is then: $J^{\mu} = (c\rho, J_x, J_y, J_z)^T$

Continuity equation become: $\partial_{\mu}J^{\mu} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

Summary

Maxwell's equations

Maxwell's Equations are:

1. Gauss: $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$
2. Ampère: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$
3. Faraday: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$
4. Anti-Dirac: $\vec{\nabla}\cdot\vec{B} = 0$

1. Gauss: $\nabla^2 V = -\rho/\epsilon_0$
2. Ampère: $\nabla^2\vec{A} - \mu_0\epsilon_0(\partial^2 A^2/\partial t^2) = \mu_0\epsilon_0\vec{\nabla}(\partial V/\partial t) - \mu_0\vec{J}$
3. Faraday: $\vec{E} = -\vec{\nabla}V - (\partial\vec{A}/\partial t)$
4. Anti-Dirac: $\vec{B} = \vec{\nabla}\times\vec{A}$

Gauge Transformation

A Gauge transformation is thus any transformation that adds $\vec{\nabla}f$ to $\vec{A}$ and adds $k(t) - \frac{\partial f}{\partial t}$ to V

In 4D Spacetime: $A^\mu \longmapsto A^\mu + \partial^\mu f$

Lorentz Gauge: $\partial_\mu A^\mu = \frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

Field Transformations

Let's consider an electric field $\vec{E}$ and a magnetic field $\vec{B}$ in 3-Dimensions

As we boost from a frame to another we have the following transformations:

$\vec{E}'_\parallel = \vec{E}_\parallel$ and $\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp)$

$\vec{B}'_\parallel = \vec{B}_\parallel$ and $\vec{B}'_\perp = \gamma(\vec{B}_\perp - (\vec{v}/c^2)\times\vec{E}_\perp)$

e.g. For a boost from a frame S to a frame S' with $\vec{v}_{S'} = v\hat{x}$ we have:

$E'_x = E_x$ $\quad E'_y = \gamma(E_y - vB_z)$ $\quad E'_z = \gamma(E_z + vB_y)$

$B'_x = B_x$ $\quad B'_y = \gamma(B_y + (v/c^2)E_z)$ $\quad B'_z = \gamma(B_z - (v/c^2)E_y)$

Field Tensor and Potentials

The 4-Potential is: $A^\mu = (V/c, \vec{A})^T$

Field Tensor is $F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$ Dual Field Tensor: $F^{\mu\nu} \longmapsto {}^*F^{\mu\nu}$ if $\vec{E}/c \longmapsto \vec{B}$, $\vec{B} \longrightarrow -\vec{E}/c$

Covariant Quantities

4-Current J^μ

Continuity equation: $\partial_\mu J^\mu(x^\mu) = 0$ where $J^\mu = (c\rho, \vec{J})$ and $\partial_\mu = \left(\frac{\partial}{\partial(ct)}, \vec{\nabla}\right)$

Justification of J^μ:

Consider $\delta q = \rho\, d^3x$ is total charge in a volume d^3x in an inertial reference frame K

Charge δq is Lorentz invariant i.e. $\rho\, d^3x = \rho' d^3x'$

Also: $d^4x = dx^0 d^3x = c\,dt\,d^3x$ is Lorentz Inv. $\Longrightarrow$ $c\rho^0$ transforms like x^0

4-Potential and Field Tensor

$\vec{E}$ and $\vec{B}$ are invariant vectors according to Galilean Group but not w.r.t. Lorentz Group

For Lorentz Group: Field Strength Tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ (~"4D-curl") 2^{nd} rank, antisymmetric and contravariant tensor

Each index of the Tensor transforms independently

$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} F^{\rho\sigma}$

$F' = \Lambda F \Lambda^T$

$A^{0'} = \gamma(A^0 - \vec{\beta}\cdot\vec{A})$

$A'_\parallel = \gamma(A_\parallel - \beta A^0)$

$\vec{A}'_\perp = \vec{A}_\perp$

Maxwell's equations

In Tensor notation:

① $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$ $\equiv$ Gauss + Ampère-Maxwell

② $\partial_\nu G^{\mu\nu} = 0$ $\equiv$ Faraday + Anti-Dirac

Maxwell's equations:

① $\Box^2 A^\mu = -\mu_0 J^\mu$ where $\Box^2 = \partial_\mu\partial^\mu$ and $\partial^\mu = g^{\mu\nu}\partial_\nu$, $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

② Automatically satisfied

Lecture 26.02.2024

Mechanics

Types of mechanics:

1) Newtonian Formulation: $\vec{F} = m\vec{a}$
2) Hamiltonian Formulation: $H = T + V$
   ↳ generalized coordinates and momenta
3) Lagrangian Formulation: $L = T - V$
   ↳ Relativistic
   ↳ $\text{Amplitude} = \int d(\text{path})\, e^{i\hbar^{-1}\text{Action}}$ Feynman Path Integral

Lagrangian Formalism

Action: $S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$

Principle of Least Action: Action is extremum/minimised i.e. $\delta S = 0$

Action is minimized if Lagrangian satisfies Euler-Lagrang Equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$

Example

Relativistic point particle with charge q in Electric and magnetic fields

$L = T - V = \frac{1}{2}mv^2 - q(V - \vec{v}\cdot\vec{A}) \Longrightarrow \vec{F} = m\vec{a} = q(\vec{E} + \vec{v}\times\vec{B})$

Fields

$\vec{E}(\vec{x}, t), \vec{B}(\vec{x}, t) \Longrightarrow$ In fields, $\vec{x}$ and t are parameters and as fields are continuous there are infinite degrees of freedom

Particles: Discrete $\Longrightarrow$ Degrees of freedom

Fields: continuous $\Longrightarrow$ Degrees of freedom are infinite

$$\left.\begin{aligned} i &\longrightarrow x^\mu, k \\ q_i &\longrightarrow \phi_k(x) \\ \dot{q}_i &\longrightarrow \partial^\mu \phi_k(x) \end{aligned}\right| \Longrightarrow L = \sum_i^n L(q_i, \dot{q}_i) \longrightarrow L = \int \underbrace{\mathcal{L}}_{\text{"Lagrangian Density"}}(\phi_k, \partial^\mu\phi_k)\, d^3x$$

Action Integral: $S = \int \mathcal{L}\, d^3x\, dt = \frac{1}{c}\int \mathcal{L}\, \underbrace{d^4x}_{\text{inv}}$ For action to be invariant we need $\mathcal{L}$ inv. If action is invariant E.O.M will be Lorentz inv.

Electromagnetism

In the case of Electromagnetism: $\phi_k \longrightarrow A^\mu$, $\partial^\mu\phi_k \longrightarrow \partial^\nu A^\mu$

We have to guess Lagrangian, which we expect it to be:

- Quadratic in velocities $\partial^\mu A^\nu$ or $F^{\mu\nu}$
- Scalar and Invariant

The only options are: $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}G^{\mu\nu}$ and $G_{\mu\nu}G^{\mu\nu}$ but $F_{\mu\nu}G^{\mu\nu}$ is a "pseudoscalar" i.e. odd under space inversion

We also need to consider source densities $J^\mu(x) = (c\rho, \vec{J})$ which bring about interactions

Postulate Lagrangian Density as

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Free Lagrangian}} + \underbrace{J_\mu A^\mu}_{\text{Interaction Lagrangian i.e. interaction of current with potential}}$$

We now have to show that Maxwell's covariant equation are directly derived from the lagrangian

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Longrightarrow F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

Solution

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{\mu_0} F_{\beta\alpha} \qquad \frac{\partial \mathcal{L}}{\partial A^\alpha} = J_\alpha \longrightarrow \partial_\beta F^{\beta\alpha} = -\mu_0 J^\alpha$$

$$\partial_\beta G^{\beta\alpha} = 0 \longleftarrow \text{ok because we use } A^\mu$$

Photon

Mass zero of photon comes from gauge invariance of theory

↳

We can only put better and better upper limit on mass of photon but never prove it is zero! Current limit: $m_\gamma c^2 < 10^{-18}$ eV

If photon had mass different frequencies would have different speed and potentials would have a fall-off

Lecture 29.02.2024

Maxwell's equations are:

- Lorenz Invariant
- Gauge Invariant $\Longrightarrow$ Leads to zero mass of photon

Lagrangian Density

$$\mathcal{L}(x) = \underbrace{-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}}_{\text{Free Term}} \underbrace{- \frac{1}{2}\mu^2 A_\alpha A^\alpha}_{\text{Mass Term}} + \underbrace{J_\alpha A^\alpha}_{\text{Interaction Term}}$$

Proca Equation

If we don't include the mass term we get: $\partial_\beta F^{\alpha\beta} = -\mu_0 J^\alpha$ from E-L Equations $\qquad \mu = m_\gamma (c/\hbar) \qquad \mu^{-1} \equiv$ "Compton Wavelength"

However this changes when we introduce the mass term: $\Box^2 A^\alpha - \mu^2 A^\alpha = -\mu_0 J^\alpha$

↳ With mass term there is no more Gauge Invariance except for Lorenz Gauge

↳ Maxwell's equations change: $\vec{\nabla} \cdot \vec{E} = -\frac{\rho}{\varepsilon_0} - \mu^2 V \Longrightarrow$ Potentials appear explicitely and thus no Gauge Inv.

↳ Adding mass leads to 3 degrees of freedom instead of 2

Aside of Lecture 04.03.2024

Charge, Energy and Momentum

Charge

The continuity equation (which can be derived from the Lagrangian or Maxwell's equations) states the local conservation of charge:

$\partial_\mu J^\mu = \partial_t \rho - \vec{\nabla}\cdot\vec{J} = 0 \implies$ A change in charge density corresponds to a current

Energy

The work done on a charge configuration is given by: $dW = \vec{F}\cdot d\vec{l}$

For a point charge: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ and $d\vec{l} = \vec{v}dt \implies dW = \vec{F}\cdot d\vec{l} = q(\vec{v}\cdot\vec{E})dt$

For a charge distribution:

$q \longmapsto \rho\, d^3x$ and $\rho\vec{v} \longmapsto \vec{J}$

$dW = (\vec{E}\cdot\vec{J})\, d^3x\, dt$ or $\dot{W} = \int (\vec{E}\cdot\vec{J})\, d^3x$

By Maxwell's equations:

$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$

$E^2 = \vec{E}\cdot\vec{E} \implies \vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$

$B^2 = \vec{B}\cdot\vec{B} \implies \vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2)$

$$\vec{E}\cdot\vec{J} = \mu_0^{-1}\left[\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \frac{1}{c^2}\vec{E}\frac{\partial\vec{E}}{\partial t}\right] = -\frac{1}{\mu_0}\left[\frac{1}{2}\partial_t\left(B^2 + \frac{1}{c^2}E^2\right) - \vec{\nabla}\cdot(\vec{E}\times\vec{B})\right] = -\partial_t u - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

It follows that:

$\dot{W} = -\int\frac{\partial u}{\partial t}\, d^3x - \underbrace{\oint \vec{S}\cdot d\vec{a}}_{\text{Power radiated away}}$

Flux density: Flux through $d\vec{a}$

Flux: Integral of Flux density over surface

Energy (density) stored in fields $\vec{E}$ and $\vec{B}$ $:= u = \frac{1}{2\mu_0}\left[B^2 + \frac{1}{c^2}E^2\right]$ $\quad [u] = J\,m^{-3}$

Energy flux density of energy leaving surface $:= \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$ $\quad [\vec{S}] = J\,m^{-2}s^{-1}$

The rate of change of work on a charge distribution is minus the change in energy stored in the fields less the energy leaving the system

If there is no charge distribution to act upon: $\dot{W} = 0$ and $\partial_t u + \vec{\nabla}\cdot\vec{S} = 0$

Momentum

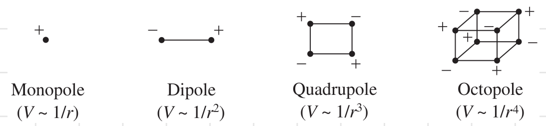
Lecture 04.03.2024

2m - poles Potentials

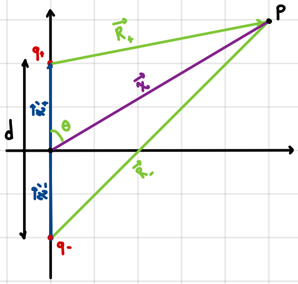
When we are far away from a charge distribution with total charge Q we can see that the electrostatic potential goes as $\sim Q/R$

Nonetheless there are still some corrections, which become apparent when Q is zero

These corrections are known as 2m-pole corrections



Example: Electrostatic dipole



$\vec{R}_+ = \vec{x} - \vec{x}'_+ \quad \vec{R}_- = \vec{x} - \vec{x}'_-$

$$R_+^2 = |\vec{x} - \vec{x}'_+|^2 = (\vec{x}-\vec{x}'_+)\cdot(\vec{x}-\vec{x}'_+) = |\vec{x}|^2 + |\vec{x}'_+|^2 - 2(\vec{x}\cdot\vec{x}') = |\vec{x}|^2 + (d/2)^2 - |\vec{x}|d\cos\theta = |\vec{x}|^2\left(1 + \frac{d^2}{4|\vec{x}|^2} - \frac{d}{|\vec{x}|}\cos\theta\right)$$

$$R_-^2 = |\vec{x} - \vec{x}'_-|^2 = (\vec{x}-\vec{x}'_-)\cdot(\vec{x}-\vec{x}'_-) = |\vec{x}|^2 + |\vec{x}'_-|^2 - 2(\vec{x}\cdot\vec{x}'_-) = |\vec{x}|^2 + (d/2)^2 + |\vec{x}|d\cos\theta = |\vec{x}|^2\left(1 + \frac{d^2}{4|\vec{x}|^2} + \frac{d}{|\vec{x}|}\cos\theta\right)$$

If $|\vec{x}| \gg d$, $R_\pm \cong |\vec{x}|\left(1 \mp \frac{d}{|\vec{x}|}\cos\theta\right)^{1/2} \Longrightarrow R_\pm^{-1} = \frac{1}{|\vec{x}|}\left(1 \mp \frac{d}{|\vec{x}|}\cos\theta\right)^{-1/2} \approx \frac{1}{|\vec{x}|}\left(1 \pm \frac{d}{2|\vec{x}|}\cos\theta\right)$

The potential is: $V(\vec{x}) = V_+(\vec{x}) + V_-(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right) = \frac{q}{4\pi\varepsilon_0}\frac{d}{|\vec{x}|^2}\cos\theta \sim \frac{1}{|\vec{x}|^2}$

The dipole potential is then: $V_{dip}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{x}}{|\vec{x}|^2}$ where $\vec{p}$ is the dipole moment $\vec{p} = qd\hat{x}'$

Radiation

Accelerating charges carry energy away to infinity, leading to radiation

Radiation: Transport of energy irreversibly out of infinity (i.e. away to infinity) by Fields

$\longrightarrow$ As such, fields are characterised by a non-zero Poynting vector

The power carried away by the fields is: $P_{rad}(t_0) = \lim_{r\to\infty} P(r,t)$

t_0 is the time at which radiation left the source and t is the current time. It follows that: $t_0 = t - |\vec{x}-\vec{x}'|/c$ and $t = t_0 + |\vec{x}-\vec{x}'|/c$

$P(r,t)$ is the Power traversing a surface a distance r away from the origin at the current time t and thus corresponds to the power radiated at t_0

As we consider $r = |\vec{x}| \longrightarrow \infty$, we can use a spherical with $A = 4\pi r^2$ and $|\vec{x} - \vec{x}'| \approx |\vec{x}| = r$

Therefore, the Poynting vector must go $\sim 1/r^2$ (at most) at larger r as if it went as r^{-m} $(m>2)$, $\vec{S}\cdot d\vec{a} \sim r^{2-m} \longmapsto 0$ as $r \longmapsto \infty$ i.e. no radiation

The multipole expansion

We now wish to find these corrections for the exact solutions to the Maxwell's equations: The retarded potentials

The potential vector can be written as: $\boxed{A^\mu(x) = \frac{\mu_0}{4\pi}\int \frac{[J^\mu(x')]_{ret}}{|\vec{x}-\vec{x}'|}d^3x'}$

We can expand this potential by considering the long wavelem the approximation by which $|\vec{x}| \gg |\vec{x}'|$

Considering: $|\vec{x}-\vec{x}'|^2 = (\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}') = |\vec{x}|^2 + |\vec{x}'|^2 - 2(\vec{x}\cdot\vec{x}') = |\vec{x}|^2\left[1 + (|\vec{x}'|/|\vec{x}|)^2 - 2(\hat{x}\cdot\vec{x}')/|\vec{x}|\right]$

Applying $|\vec{x}'| \ll |\vec{x}|$: $|\vec{x}-\vec{x}'|^2 \simeq |\vec{x}|^2\left(1 - 2(\hat{x}\cdot\vec{x}')/|\vec{x}|\right)$ or

$$|\vec{x}-\vec{x}'| \simeq |\vec{x}|\left(1 - 2(\hat{x}\cdot\vec{x}')/|\vec{x}|\right)^{1/2} \approx |\vec{x}|\left(1 - \frac{(\hat{x}\cdot\vec{x}')}{|\vec{x}|} + \frac{1}{4}\frac{(\hat{x}\cdot\vec{x}')^2}{|\vec{x}|^2} + \dots\right)$$

It follows that: $|\vec{x}-\vec{x}'|^{-1} \simeq |\vec{x}|^{-1}(1+\varepsilon)^{-1/2} \approx |\vec{x}|^{-1}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \dots\right) = \frac{1}{|\vec{x}|^{-1}}\left[1 + \frac{(\hat{x}\cdot\vec{x}')}{|\vec{x}|} + \frac{3}{2}\frac{(\hat{x}\cdot\vec{x}')^2}{|\vec{x}|^2} + \frac{5}{2}\frac{(\hat{x}\cdot\vec{x}')^3}{|\vec{x}|^3} + \dots\right]$

We can thus write: $\boxed{A^\mu(x) = \frac{\mu_0}{4\pi}\sum_{m=0}^{\infty}\frac{1}{|\vec{x}|^{m+1}}\underbrace{\int(\hat{x}\cdot\vec{x}')^m[J^\mu(x')]_{ret}\,d^3x'}_{2m\text{-pole term}}} \approx \frac{\mu_0}{4\pi|\vec{x}|}\int[J^\mu(x')]_{ret}d^3x' + \frac{\mu_0}{4\pi|\vec{x}|^2}\int(\hat{x}\cdot\vec{x}')[J^\mu(x)]_{ret}\,d^3x'$

Dipole Radiation

By making retarded time dependence explicit we have: $[J^\mu(x')]_{ret} = J^\mu(\vec{x}', t_{ret})$ where $t_{ret} = t - |\vec{x}-\vec{x}'|c^{-1} \approx t - |\vec{x}|c^{-1} + (\hat{x}\cdot\vec{x}')c^{-1}$

If it changes slowly in time we can expand $J^\mu(\vec{x}', t_{ret})$ around the retarded time at the origin i.e. $t_0 = t - |\vec{x}|/c$

Then:

$$J^\mu(\vec{x}', t - c^{-1}|\vec{x}-\vec{x}'|) \approx J^\mu(\vec{x}', t_0) + \dot{J}^\mu(\vec{x}', t_0)(t_{ret}-t_0) + \frac{1}{2}\ddot{J}^\mu(\vec{x}', t_0)(t_{ret}-t_0) + \ldots = J^\mu(\vec{x}', t_0) + \dot{J}^\mu(\vec{x}', t_0)(\hat{x}\cdot\vec{x}')c^{-1} + \frac{1}{2}\ddot{J}^\mu(\vec{x}', t_0)(\hat{x}\cdot\vec{x}')^2 c^{-2} + \ldots$$

As it changes slowly we have: $J^\mu(\vec{x}', t_{ret}) \approx J^\mu(\vec{x}', t_0) + \dot{J}^\mu(\vec{x}', t_0)(\hat{x}\cdot\vec{x}')c^{-1}$

It follows that: $A^\mu(x) \simeq \frac{\mu_0}{4\pi}\sum_{m=0}^{\infty}\frac{1}{|\vec{x}|^{m+1}}\int d^3x'(\hat{x}\cdot\vec{x}')^m\left[J^\mu(\vec{x}', t_0) + \frac{(\hat{x}\cdot\vec{x}')}{c}\dot{J}^\mu(\vec{x}', t_0)\right] =$

$$\simeq \frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{x}|}\int J^\mu(\vec{x}', t_0)d^3x' + \frac{1}{c|\vec{x}|}\int(\hat{x}\cdot\vec{x}')\dot{J}^\mu(\vec{x}', t_0)d^3x' + \frac{1}{|\vec{x}|^2}\int(\hat{x}\cdot\vec{x}')J^\mu(\vec{x}', t_0)d^3x' + \frac{1}{c|\vec{x}|^2}\int(\hat{x}\cdot x')^2\dot{J}(\vec{x}', t_0)d^3x' + \ldots\right\}$$

$$\approx \frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{x}|}\int J^\mu(\vec{x}', t_0)d^3x' + \frac{1}{c|\vec{x}|}\int(\hat{x}\cdot\vec{x}')\dot{J}^\mu(\vec{x}', t_0)d^3x' + \frac{1}{|\vec{x}|^2}\int(\hat{x}\cdot\vec{x}')J^\mu(\vec{x}', t_0)d^3x'\right\} \quad \text{As } (\hat{x}\cdot\vec{x}')^2/(c|\vec{x}|^2) \approx 0$$

Then: $V(\vec{x}, t) = cA^0(x) \approx \frac{\mu_0 c^2}{4\pi}\left\{\frac{1}{|\vec{x}|}\int\rho(\vec{x}', t_0)d^3x' + \frac{\hat{x}}{c|\vec{x}|}\cdot\int\vec{x}'\dot{\rho}(\vec{x}', t_0)d^3x' + \frac{\hat{x}}{|\vec{x}|^2}\cdot\int\vec{x}'\rho(\vec{x}', t_0)d^3x'\right\} = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{|\vec{x}|} + \frac{\hat{x}\cdot\vec{p}_0}{|\vec{x}|^2} + \frac{\hat{x}\cdot\dot{\vec{p}}_0}{c|\vec{x}|}\right\}$

$\vec{A}(\vec{x}, t) \approx \frac{\mu_0}{4\pi|\vec{x}|}\int\vec{J}(\vec{x}', t_0)d^3x' = \frac{\mu_0}{4\pi|\vec{x}|}\dot{\vec{p}}_0$ (Why can we ignore the next terms?)

Why is $\dot{\vec{p}}_0 = \int d^3x'\,\vec{J}(\vec{x}', t_0)$?

$\vec{p} = \int\vec{x}'\rho(\vec{x}', t)d^3x' \implies \dot{\vec{p}} = \int\vec{x}'\dot{\rho}(\vec{x}', t)d^3x' = -\int\vec{x}'(\vec{\nabla}\cdot\vec{J})d^3x'$

$\vec{\nabla}(x^i\vec{J}) = x^i(\vec{\nabla}\cdot\vec{J}) + \vec{J}\cdot(\vec{\nabla}x^i) = x^i(\vec{\nabla}\cdot\vec{J}) + J^i \implies \vec{J} - (\hat{x}')^i\,\vec{\nabla}\cdot(x^i\vec{J}) = -(x')^i(\vec{\nabla}\cdot\vec{J})(\hat{x}')^i = -\vec{x}(\vec{\nabla}\cdot\vec{J})$

$\dot{\vec{p}} = \int d^3x'\,J(\vec{x}', t) - (\hat{x}'_i)\int d^3x'\,\vec{\nabla}\cdot(x^i\vec{J}) = \int d^3x'\,J(\vec{x}', t) - (\hat{x}')^i\oint_S (x')^i\vec{J}\cdot d^2\vec{x}'$

$\longrightarrow 0$ as $\vec{J}$ is entirely in the enclosed volume

Electric and magnetic field

Maxwell's equations: $\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t \qquad \vec{B} = \vec{\nabla}\times\vec{A}$

We are interested in terms that contribute to the radiated power i.e. terms at most $\sim 1/|\vec{x}|$ as $|\vec{x}|^k$ $(k\geq 2)$ lead to $\vec{S}\sim r^{-m}$, $m>2$ and $P\longrightarrow 0$ as $|\vec{x}|\longrightarrow\infty$

$$\vec{\nabla}V = \frac{1}{4\pi\epsilon_0}\left[Q\vec{\nabla}\left(\frac{1}{|\vec{x}|}\right) + \vec{\nabla}(\hat{x}\cdot\vec{p}_0)|\vec{x}|^{-2} + (\hat{x}\cdot\vec{p}_0)\vec{\nabla}\left(\frac{1}{|\vec{x}|^2}\right) + \frac{\vec{\nabla}(\hat{x}\cdot\dot{\vec{p}}_0)}{c|\vec{x}|} + \frac{(\hat{x}\cdot\dot{\vec{p}}_0)}{c}\vec{\nabla}\left(\frac{1}{|\vec{x}|}\right)\right] =$$

$$= \frac{1}{4\pi\epsilon_0}\left[-\frac{Q}{|\vec{x}|^2}\hat{x} + \frac{\vec{\nabla}(\hat{x}\cdot\vec{p}_0)}{|\vec{x}|^2} - 2\frac{(\hat{x}\cdot\vec{p}_0)}{|\vec{x}|^3} + \frac{\vec{\nabla}(\hat{x}\cdot\dot{\vec{p}}_0)}{c|\vec{x}|} - \frac{(\hat{x}\cdot\dot{\vec{p}}_0)}{c|\vec{x}|^2}\right] =$$

$$\approx \frac{1}{4\pi\epsilon_0}\frac{\vec{\nabla}(\hat{x}\cdot\dot{\vec{p}}_0)}{c|\vec{x}|} = \frac{1}{4\pi\epsilon_0}\frac{(\hat{x}\cdot\vec{\nabla})\dot{\vec{p}}_0 + (\dot{\vec{p}}_0\cdot\vec{\nabla})\hat{x} + \hat{x}\times(\vec{\nabla}\times\dot{\vec{p}}_0) + \dot{\vec{p}}_0\times(\vec{\nabla}\times\hat{x})}{c|\vec{x}|} = \frac{(\hat{x}\cdot\ddot{\vec{p}}_0)}{4\pi\epsilon_0 c|\vec{x}|}(\vec{\nabla}t_0) = -\frac{\hat{x}\cdot\ddot{\vec{p}}_0}{4\pi\epsilon_0 c^2|\vec{x}|}\hat{x}$$

$$\partial\vec{A}/\partial t = \frac{1}{4\pi\epsilon_0|\vec{x}|}\frac{\ddot{\vec{p}}_0}{c^2} = \frac{\mu_0}{4\pi|\vec{x}|}\ddot{\vec{p}}$$

$$\vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\times\left(\frac{\dot{\vec{p}}_0}{|\vec{x}|}\right)\right] = \frac{\mu_0}{4\pi|\vec{x}|}(\vec{\nabla}\times\dot{\vec{p}}_0) + \frac{\mu_0}{4\pi}(\vec{\nabla}|\vec{x}|^{-1})\times\dot{\vec{p}}_0 \approx \frac{\mu_0}{4\pi|\vec{x}|}\int\vec{\nabla}'\times(\vec{x}'\dot{\rho}(\vec{x}', t_0))d^3x' =$$

$$= \frac{\mu_0}{4\pi|\vec{x}|}\int(\vec{\nabla}\dot{\rho}(\vec{x}', t_0))\times\vec{x}'d^3x' = \frac{\mu_0}{4\pi|\vec{x}|}\int(\vec{\nabla}(t_0)\times\ddot{\rho}(\vec{x}', t_0)\vec{x}')d^3x' = -\frac{\mu_0}{4\pi c|\vec{x}|}(\hat{x}\times\ddot{\vec{p}}_0)$$

Therefore, the fields are:

$$\vec{E}(\vec{x}, t) = \frac{\mu_0}{4\pi|\vec{x}|}\left[(\hat{x}\cdot\ddot{\vec{p}})\hat{x} - \ddot{\vec{p}}\right]_{ret} = \frac{\mu_0}{4\pi|\vec{x}|}\left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})\right]_{ret} \quad \text{where } t_{ret}\approx t_0 = t - |\vec{x}|/c$$

$$\vec{B}(\vec{x}, t) = -\frac{\mu_0}{4\pi c|\vec{x}|}(\hat{x}\times\ddot{\vec{p}})_{ret}$$

Poynting Vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = -\frac{\mu_0}{16\pi^2 c|\vec{x}|^2}\left[(\hat{x}\cdot\ddot{\vec{p}})(\hat{x}\times(\hat{x}\times\ddot{\vec{p}})) - (\ddot{\vec{p}}\times(\hat{x}\times\ddot{\vec{p}}))\right]_{ret} =$$

$$= -\frac{\mu_0}{16\pi^2 c|\vec{x}|^2}\left[(\hat{x}\cdot\ddot{\vec{p}})\left[\hat{x}(\hat{x}\cdot\ddot{\vec{p}}) - \ddot{\vec{p}}(\hat{x}\cdot\hat{x})\right] - \left[\hat{x}(\ddot{\vec{p}}\cdot\ddot{\vec{p}}) - \ddot{\vec{p}}(\ddot{\vec{p}}\cdot\hat{x})\right]\right]_{ret}$$

$$= \frac{\mu_0}{16\pi^2 c|\vec{x}|^2}\left[|\ddot{\vec{p}}|^2\hat{x} + \ddot{\vec{p}}(\hat{x}\cdot\ddot{\vec{p}}) - (\hat{x}\cdot\ddot{\vec{p}})(\hat{x}\cdot\ddot{\vec{p}})\hat{x} - \ddot{\vec{p}}(\ddot{\vec{p}}\cdot\hat{x})\right]_{ret} = \frac{\mu_0}{16\pi^2 c|\vec{x}|^2}\left[|\ddot{\vec{p}}|^2 - (\hat{x}\cdot\ddot{\vec{p}})^2\right]\hat{x}$$

Power Radiated

The Poynting vector is given by: $\vec{S} = \frac{\mu_0}{16\pi^2 c|\vec{\imath}|^2}\left[|\ddot{\vec{p}}|^2 - (\hat{\imath}\cdot\ddot{\vec{p}})^2\right]\hat{\imath}$

The power radiated out of a sphere of radius R is: $P(R,t) = \oint \vec{S}(R,t)\cdot d\vec{a}$

Choosing the z-direction to correspond to the direction of $\ddot{\vec{p}}$ we have:

$$\left.\begin{aligned} d\vec{a} &= R^2 \sin\theta \, d\theta \, d\phi \, \hat{\imath} \\ \vec{S} &= \frac{\mu_0}{16\pi^2 c|\vec{\imath}|^2}\left[|\ddot{\vec{p}}|^2 \sin^2\theta\right]_{ret}\hat{\imath} \\ \vec{S}\cdot d\vec{a} &= \frac{\mu_0}{16\pi^2 c}|\ddot{\vec{p}}(t_0)|^2 \sin^3\theta \end{aligned}\right| \Longrightarrow \quad P(R,t) = \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}(t_0)|^2$$

Larmor Formula for E1 (Dipole) Radiation: $P(R,t) = \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}(t_0)|^2$

If dipole ≈ 0, further corrections by magnetic dipole μ and by electric quadrupole Q

Monopole does not radiate as charge is conserved $\Longrightarrow$ Conservation of charge inhibits changes in monopole field strengths over time

It is also interesting to look at the direction in which the particle radiates the most strongly

We have: $P = \int \vec{S}\cdot d\vec{a} = \int \frac{dP}{d\Omega} d\Omega$ from which follows that $\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}|\ddot{\vec{p}}(t_0)|^2 \sin^3\theta$

A particle radiates the most along a direction perpendicular to its dipole moment

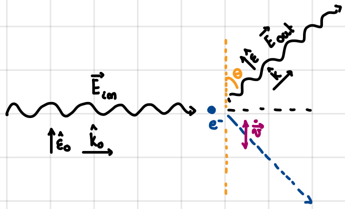
Example: Larmor Formula for a Point Particle

Thomson Scattering

Consider the plane wave $\vec{E}_0 = \hat{\varepsilon}_0 E_0 e^{i(\vec{k}_0 \cdot \vec{x} - \omega_0 t)}$ polarised along $\hat{\varepsilon}_0$ and propagating along $\hat{k}_0$

Now consider an electron on which the wave scatters

The scattered wave will be: $\vec{E} = \hat{\varepsilon} E e^{i(\vec{k} \cdot \vec{x} - \omega t)}$



The force on e^- is given by: $\vec{F} = m\dot{\vec{v}} = q(\vec{E}_{in} + \vec{v} \times \vec{B}_{in}) \approx -e\vec{E}_{in}$

The dipole moment is given by: $\vec{p} = q\vec{r} \Longrightarrow \ddot{\vec{p}} = q\ddot{\vec{r}}$ i.e. $\ddot{\vec{p}} = q\dot{\vec{v}} = \frac{q^2}{m}\vec{E}_{in}$

Therefore, the scattered electron will accelerate along the direction of oscillation of the incident electric field and it will radiate along a direction $\hat{k}$ at angle θ wrt dipole

By the Larmor Formula for E1 (Dipole) Radiation we have: $\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\left[|\ddot{\vec{p}}(t_0)|\sin\theta\right]^2 = \frac{\mu_0}{16\pi^2 c}|\hat{\varepsilon}^* \cdot \ddot{\vec{p}}(t_0)|^2 = \frac{e^2\mu_0}{16\pi^2 c}|\hat{\varepsilon}^* \cdot \dot{\vec{v}}|^2$

The average over one cycle is:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 \mu_0}{16\pi^2 m^2 c}\langle |\hat{\varepsilon} \cdot \vec{E}_{in}|^2 \rangle = \frac{\mu_0 c^4}{32\pi^2 c}\left(\frac{e^2}{mc^2}\right)^2 |E_0|^2 \left|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\text{energy radiated/unit time/unit solid angle}}{\text{incident energy flux/unit area/unit time}} \Longrightarrow \frac{d\sigma}{d\Omega} = \frac{\mu_0 c^3}{32\pi^2} 2\mu_0 c \left(\frac{e^2}{mc^2}\right)^2 |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2 = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2 \Longrightarrow \sigma_T = \frac{8\pi}{3} r_c^2$$

→ Time averaged Poynting vector for Plane Wave: $\langle \vec{S} \rangle = |\vec{E}_0|^2 / (2\mu_0 c)$

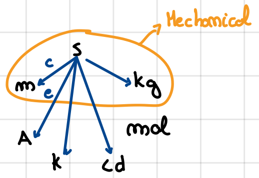
Lecture 11.03.2024

Dimensional Analysis of the Atomic Bomb

Fine Structure Constant: α

- In SI (kg, m, s): $\alpha = e^2/(4\pi\varepsilon_0 \hbar c) \cong 1/137$ In all units $\alpha \simeq 1/137$
- In "Gaussian" (g, cm, s): $\alpha = e^2/(\hbar c) \cong 1/137$
- In "Heaviside-Lorentz": $\alpha = e^2/(4\pi\hbar c) \cong 1/137$

S.I. Units:



$\Longrightarrow$ These units can all be derived from constants

- s from $\Delta\nu_{Cs}$
- kg from h
- K from k_B
- mol from N_A
- m from c
- cd from K_{cd}
- A from e

This relation hold as long as SR and QM are assumed valid

Only m, s, kg are fundamental

Coulomb Law: ε_0 acts as conversion factor between mechanical units

$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$ S.I.

$F = \frac{q_1 q_2}{r^2}$ Gauss No $\varepsilon_0 \longrightarrow$ Charge is already in mechanical units $\longrightarrow$ charge not fundamental

$F = \frac{q_1 q_2}{4\pi r^2}$ HL

Natural units: $c \longrightarrow 1$, $\hbar \longrightarrow 1$ (Not $c = \hbar = 1$ as c and $\hbar$ have diff. dimension) $\Longrightarrow$ Velocities in units of c, Ang. Mom. in units of $\hbar$

Particle Physics units: Everything in units of length or energy

Aharonov-Bohm Effect

$\vec{E}$ and $\vec{B}$ are not fundamental as they are macrocoscopic manifestations of charge distribution

However, $\vec{E}$ and $\vec{B}$ are observables while the more fundamental quantity A^μ is not due to Gauge Freedom i.e. do not appear explicitly in observable quantities

In QM the fundamental quantity are the photons, the quanta of the field A^μ

Hamiltonian: $H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + e\vec{A}\right)^2$ where $\vec{p} \longrightarrow \vec{p} - q\vec{A}$ with $q = -e$ and we ignore V

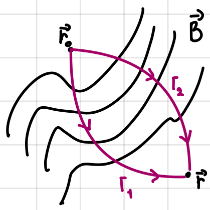
$H\varphi_0(r) = E\,\varphi_0(r)$

$\varphi(\vec{r}) = e^{ie\Lambda(\vec{r})}\varphi_0(\vec{r})$

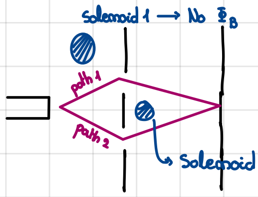
Choose Λ s.t. $e\hbar\vec{\nabla}\Lambda + e\vec{A} = 0$ (i.e. Gauge Invariance) s.t. $H\varphi(\vec{r}) = E\varphi(\vec{r})$

Solve $\Lambda(\vec{r}) = -\frac{1}{\hbar}\int_{\vec{r}_0}^{\vec{r}} d\vec{l}' \cdot \vec{A}(\vec{r}')$

$\varphi(\vec{r}) = e^{-i\frac{e}{\hbar}\int_{\vec{r}_0}^{\vec{r}} d\vec{l}'\cdot\vec{A}(\vec{r}')}\varphi_0(\vec{r})$ Is it independent of path?



$$\int_{\Gamma_1} d\vec{l}\cdot\vec{A} - \int_{\Gamma_2} d\vec{l}\cdot\vec{A} = \oint d\vec{l}\cdot\vec{A} = \iint d\vec{S}\cdot(\vec{\nabla}\times\vec{A}) = \iint d\vec{S}\cdot\vec{B} = \Phi_B$$



Solenoid 2 $\longrightarrow$ $\Phi_B \neq 0$ but e^- does interact with $\vec{B}$ as it doesn't get inside solenoid

No Solenoid $\varphi = \varphi_1 + \varphi_2 = R_1 e^{iS_1} + R_2 e^{iS_2} = e^{iS_1}\left(R_1 + e^{i(S_2 - S_1)}R_2\right)$

Solenoid 2 $\varphi = e^{-\frac{ie}{\hbar}\int d\vec{l}'\cdot\vec{A}(\vec{r}')}\varphi_1 + e^{-\frac{ie}{\hbar}\int d\vec{l}'\cdot\vec{A}(\vec{r}')}\varphi_2 = e^{-\frac{ie}{\hbar}\int\dots} e^{iS_1}\left(R_1 + R_2 e^{i(S_2 - S_1) + ie\Phi/\hbar}\right)$ changes in interference pattern

Aharonov - Bohm Effect

The Hamiltonian

The Lagrangian of a particle in an EM field is given by: $L = \frac{1}{2} m v_\mu v^\mu + q v_\mu A^\mu = \gamma^{-1}\left[\frac{1}{2} m(\vec{v}\cdot\vec{v}) - \frac{1}{2} mc^2\right] - \gamma q (V - \vec{v}\cdot\vec{A})$

In non-relativistic limit: $L \approx \frac{1}{2} m(\vec{v}\cdot\vec{v}) - q(V - \vec{v}\cdot\vec{A})$

The canonical momentum is: $\vec{P}_{can} = (\partial L/\partial \vec{v}) = m\vec{v} + q\vec{A} = \vec{p} + q\vec{A}$

The Hamiltonian is thus: $H = \vec{P}_{can}\cdot\vec{v} - L = \frac{1}{2} mv^2 + qV = \frac{p^2}{2m} + qV = \frac{1}{2m}(\vec{P}_{can} - q\vec{A})^2 + qV$

The quantum Hamiltonian is thus: $\hat{H} = \frac{1}{2m}(-i\hbar\vec{\nabla} - q\vec{A})\cdot(-i\hbar\vec{\nabla} - q\vec{A}) + qV$

Schrodinger Equation and Gauge Transformations

While the Hamiltonian depends only on the potentials $\vec{A}$ and V, we know tha these potentials manifest macroscopically as the fields $\vec{E}$ and $\vec{B}$

These fields are given by $\vec{E} = -\vec{\nabla}V - (\partial\vec{A}/\partial t)$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ and thus present the following gauge invariance:

$$\vec{A} \longmapsto \vec{A}' = \vec{A} + \vec{\nabla}\chi$$
$$V \longmapsto V' = V - \partial_t\chi$$

Due to a Gauge Transformation: $\hat{H} \longmapsto \hat{H}'$

$$\hat{H} = (2m)^{-1}(-i\hbar\vec{\nabla} - q\vec{A})^2 + qV = (2m)^{-1}\left[-\hbar^2\nabla^2 + q^2A^2 + iq\hbar\,(\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla})\right] + qV$$

$$\hat{H}' = (2m)^{-1}\left[-i\hbar\vec{\nabla} - q\vec{A}'\right]^2 + qV' = (2m)^{-1}\left[-\hbar^2\nabla^2 + q^2(\vec{A}'\cdot\vec{A}') + iq\hbar(\vec{\nabla}\cdot\vec{A}' + \vec{A}'\cdot\vec{\nabla})\right] + qV' =$$
$$= (2m)^{-1}\left[-\hbar^2\nabla^2 + q^2(A^2 + 2(\vec{A}\cdot\vec{\nabla}\chi) + (\vec{\nabla}\chi)^2) + iq\hbar(\vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}\chi + \vec{A}\cdot\vec{\nabla} + \vec{\nabla}\chi\cdot\vec{\nabla})\right] + qV - q\partial_t\chi =$$
$$= (2m)^{-1}\left[(-i\hbar\vec{\nabla} - q\vec{A})^2 + q^2(2\vec{A}\cdot\vec{\nabla}\chi + (\vec{\nabla}\chi)^2) + iq\hbar(\vec{\nabla}\cdot\vec{\nabla}\chi + \vec{\nabla}\chi\cdot\vec{\nabla})\right] + qV - q\partial_t\chi =$$
$$= \hat{H} + \frac{q^2}{2m}(\vec{A}\cdot\vec{\nabla}\chi + \vec{\nabla}\chi\cdot\vec{A} + (\vec{\nabla}\chi)^2) + \frac{iq\hbar}{2m}(\vec{\nabla}\cdot\vec{\nabla}\chi + \vec{\nabla}\chi\cdot\vec{\nabla}) - q\partial_t\chi$$

The Schrödinger equation is:

- Before Gauge Transformation: $i\dot{\psi} = \hat{H}\psi = \frac{1}{2m}(-i\hbar\vec{\nabla} - q\vec{A})\cdot(-i\hbar\vec{\nabla} - q\vec{A})\psi + qV\psi$
- After Gauge Transformation: $i\dot{\psi}' = \hat{H}'\psi' = \hat{H}\psi' + \frac{q^2}{2m}(\vec{A}\cdot\vec{\nabla}\chi + \vec{\nabla}\chi\cdot\vec{A} + (\vec{\nabla}\chi)^2)\psi' + \frac{iq\hbar}{2m}(\vec{\nabla}\cdot\vec{\nabla}\chi + \vec{\nabla}\chi\cdot\vec{\nabla})\psi' - q(\partial_t\chi)\psi'$

For ψ and ψ' to describe the same system, ψ' can differ to ψ by only a phase factor $\psi' = \psi e^{i\Lambda}$

It follows that:

$$\dot{\psi}' = [\dot{\psi} + i\partial_t\Lambda\,\psi]e^{i\Lambda} \quad \text{and} \quad \hat{H}\psi' = (2m)^{-1}\left[-\hbar^2\nabla^2(\psi e^{i\Lambda}) + q^2A^2\psi e^{i\Lambda} + iq\hbar\left((\vec{\nabla}\cdot\vec{A})\psi e^{i\Lambda} + \vec{A}\cdot(\vec{\nabla}\psi + \psi i\vec{\nabla}\Lambda)e^{i\Lambda}\right)\right] + qV\psi e^{i\Lambda} =$$
$$= (\hat{H}\psi)e^{i\Lambda} - \frac{\hbar^2}{2m}\left[2i(\vec{\nabla}\Lambda)\cdot(\vec{\nabla}\psi) + i(\nabla^2\Lambda)\psi - (\vec{\nabla}\Lambda)^2\psi\right]e^{i\Lambda} - \frac{q\hbar}{2m}(\vec{A}\cdot\vec{\nabla}\Lambda)\psi e^{i\Lambda} + qV\psi e^{i\Lambda}$$

$$\nabla^2(\psi e^{i\Lambda}) = \vec{\nabla}\cdot(\vec{\nabla}\psi + i(\vec{\nabla}\Lambda)\psi)e^{i\Lambda} = (\nabla^2\psi)e^{i\Lambda} + i(\nabla^2\Lambda)\psi e^{i\Lambda} + i(\vec{\nabla}\Lambda)\cdot(\vec{\nabla}\psi)e^{i\Lambda} + (\vec{\nabla}\psi + i(\vec{\nabla}\Lambda)\psi)\cdot i(\vec{\nabla}\Lambda)e^{i\Lambda} =$$
$$= \left[(\nabla^2\psi) + 2i(\vec{\nabla}\Lambda)\cdot(\vec{\nabla}\psi) + i(\nabla^2\Lambda)\psi - (\vec{\nabla}\Lambda)^2\psi\right]e^{i\Lambda}$$

Throug careful substitution and rearranging one can get:

$$i\hbar\dot{\psi} = \hat{H}\psi + \left[q(\vec{\nabla}\chi) - \hbar(\vec{\nabla}\Lambda)\right]\cdot\left[\frac{i\hbar}{m}(\vec{\nabla}\psi) + \frac{1}{2m}\left(q(\vec{\nabla}\chi) - \hbar(\vec{\nabla}\Lambda) + 2q\vec{A}\right)\psi\right] + \frac{i\hbar}{2m}(q\nabla^2\chi - \hbar^2\nabla^2\Lambda)\psi + (\hbar\partial_t\Lambda - q\partial_t\chi)\psi$$
$$= \hat{H}\psi + \vec{\nabla}(q\chi - \hbar\Lambda)\cdot\left[\frac{i\hbar}{m}(\vec{\nabla}\psi) + \frac{1}{2m}\left(q(\vec{\nabla}\chi) - \hbar(\vec{\nabla}\Lambda) + 2q\vec{A}\right)\psi\right] + \frac{i\hbar}{2m}\left[\nabla^2(q\chi - \hbar\Lambda)\right]\psi - \partial_t(q\chi - \hbar\Lambda)\psi$$

We thus recover the original Sch. Eq. if $\hbar\Lambda - q\chi = \text{const} \implies \psi' = e^{i\Lambda}\psi$ where $\Lambda = \frac{q\chi}{\hbar} + \text{const.}$

However, as we are generally interested in phase differences $\delta = \Lambda_1 - \Lambda_2$ we can set the constant to zero

Consequence

We proved above that a gauge transformation introduces a phase term in the wavefunction i.e. $\psi' = \psi e^{i\Lambda} \iff \psi = \psi' e^{-i\Lambda}$ where $\Lambda = q\chi/\hbar$

We can thus exploit gauge transformations to simplify H' and solve for ψ'.

By multiplying ψ' by the phase factor we can then find ψ even if $\hat{H}$ is very difficult

Application

In general, the Hamiltonian can be greatly simplified by setting $\vec{A}'=0$ s.t. $\hat{H}' = -\frac{\hbar^2}{2m}\nabla^2 + qV'$

This corresponds to a phase shift $\Lambda = -\frac{q}{\hbar}\int_\Gamma \vec{A}\cdot d\vec{l}$ where Γ is the path followed by the particle between points $\vec{r}_0$ and $\vec{r}$

Proof: By Gauge Transformation:

$\vec{A} \longmapsto \vec{A}' = \vec{A} + \vec{\nabla}\chi$ Note: Gradient theorem $\int_{\vec{r}_0}^{\vec{r}} (\vec{\nabla} f)\cdot d\vec{l} = f(\vec{r}) - f(\vec{r}_0)$

$V \longmapsto V' = V - \partial_t \chi$

If $\vec{A}'=0$, it follows that: $\vec{\nabla}\chi = -\vec{A} \implies \chi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{A}\cdot d\vec{l}$ with $\chi(\vec{r}_0)=0$ i.e. $\vec{r}_0$ is any convenient reference point

Now consider a number of particles going from $\vec{r}_0$ to $\vec{r}$. The m^{th} particle follows the path Γ_m

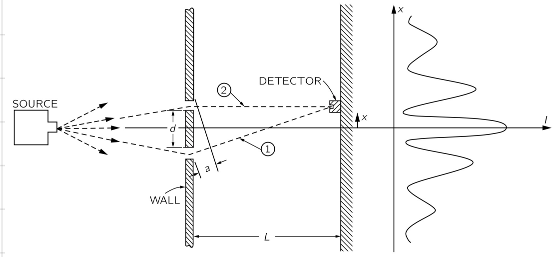
The phase different between any two particles is given by:

$$\delta = \Lambda_i - \Lambda_j = \frac{q}{\hbar}\int_{\Gamma_i - \Gamma_j} \vec{A}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_S \vec{B}\cdot d\vec{a} = \frac{q}{\hbar}\Phi_B$$

whe S is the surface enclosed by $\Gamma = \Gamma_i - \Gamma_j$ and Φ_B is the flux of B

A more accurate description would be: $\delta = \delta(B=0) + (q/\hbar)\Phi_B$ where $\delta(B=0)$ is the phase difference not due to $\vec{A}$ and $\vec{B}$

Classical VS Quantum Effect and Interpretation



Interference: Consider electrons 1 and 2 with amplitudes $c_1 e^{i\phi_1}$ and $c_2 e^{i\phi_2}$ respectively

The interference pattern will depend on the relative phase between the two electrons i.e. $\delta = \phi_1 - \phi_2$

Why?

Wavefunction: $\psi = c_1 e^{i\phi_1}\varphi + c_2 e^{i\phi_2}\varphi$

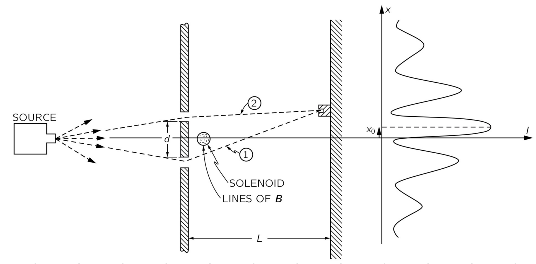
Overlap : $\langle\psi|\psi\rangle = |c_1|^2 + |c_2|^2 + c_1^* c_2 e^{-i\delta} + c_2^* c_1 e^{i\delta} \propto I$

By setting $c_1 = c_2$: $I \propto 2|c|^2(1+\cos(\delta))$

Geometrically, one can calculate $\delta = \frac{a}{\lambda} = \frac{x}{L}\frac{d}{\lambda}$

Intensity is:

- maximum when $\delta = \{0, \pm 1, \pm 2, \dots\}\pi$ i.e. constructive interference
- minimum when $\delta = \{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots\}\pi$ i.e. destructive interference



Shift in the interference pattern: Now consider the addition of a very tiny (itty-bitty) solenoid in the region enclosed by the paths of the two electrons.

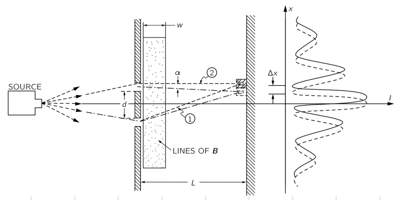
2 Situations:

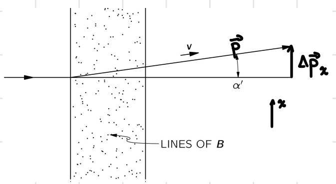
1) No current: Zero $\vec{B}$ and zero $\vec{A}$ $\implies \delta = \delta(B=0) = \phi_1 - \phi_2$

2) Current: $\vec{B}$ entirely confined within solenoid but non-zero $\vec{A}$ outside $\implies \delta = \delta(B=0) - \frac{q}{\hbar}\Phi_B$

In case (2), the phase difference is changed and thus the location of each peak is changed by

$\Delta x = -\frac{L}{d}\lambda\,\Delta\delta = -\frac{q}{\hbar}\frac{L}{d}\lambda\,\Phi_B$





Classical

$\Phi_B = Bwd$

$\vec{F} = \dot{\vec{p}} = q(\vec{E} + \vec{v}\times\vec{B})$

$\Delta\vec{p}_x = -qwB = -q\Phi_B/d$

$\alpha' = -q\Phi_B/pd = -\frac{q}{\hbar}\frac{\lambda}{d}\Phi_B$

Quantum

$\Delta x = -\frac{L}{d}\lambda\frac{q}{\hbar}\Phi_B$

$\alpha = \Delta x / L = -\frac{q\lambda}{d\hbar}\Phi_B = \alpha'$

Interpretation

From the analysis we see how it is that the vector potential which appears in quantum mechanics in an explicit form produces a classical force which depends only on its derivatives. In quantum mechanics what matters is the interference between nearby paths; it always turns out that the effects depend only on how much the field $\boldsymbol{A}$ *changes* from point to point, and therefore only on the derivatives of $\boldsymbol{A}$ and not on the value itself. Nevertheless, the vector potential $\boldsymbol{A}$

Lecture 18.03.2024

Maxwell and QM

Maxwell equations are perfect for many situations e.g. for away objects such as stars, etc.
However whenever we look at single particles systems or particle interactions, Maxwell's equations are not appropriate

To extend electrodynamics to quantum situations we will need to quantize the fields. However, the relevant fields are not the $\vec{E}$ and $\vec{B}$ field but rathe the photon field A^μ of which $\vec{E}$ and $\vec{B}$ are a macroscopic manifestation. The photon field is the fundamental quantity as seen through the Lagrangian and the "Aharonov Effect". Quantization of Electrodynamics cannot, however, take place as in standard QM as it is not relativistic i.e. $\vec{x}$ is an operator but t is just a parameter. We thus treat $\vec{x},t$ on equal footing by treating both as parameters and promoting the field A^μ to an operator (QFT)

Why fields? Fields ensure locality in classical physics (no more action at a distance) $\Longrightarrow$ Quantization of fields can be consistent with special relativity

Quantizing EM Field A^μ

Fixing the Gauge

The theory is gauge invariant $\Longrightarrow$ Properties of theory and of the field quanta must be the same in every Gauge
Two most commonly used gauges:

- Lorentz Gauge: $\partial_\mu A^\mu$
  - ↳ Lorentz Invariant
  - ↳ Simplifies most equations and Lagrangian
- Coulomb/Radiation Gauge: $\vec{\nabla}\cdot\vec{A}=0 \quad V=0$
  - ↳ Not Lorentz Invariant
  - ↳ Fixes A_0 and adds a constraint on the 3-components of $\vec{A}$ $\Longrightarrow$ Reduces degrees of freedom to two

Note: Photon has initially 4-degrees of freedom $A_0, A_\parallel, A_\perp$ where $A_\perp$ includes two. Coulomb gauge reduces it to two ($A_0=0$, $\vec{\nabla}\cdot\vec{A}=0$)
The $A_0, A_\parallel$ are called instantaneous Coulomb Potentials

Even though the theory presents gauge invariance, we must choose and fix a gauge to continue the quantization.
We choose Coulomb's gauge

Quantization of the "Free Theory" in a box

Free theory Lagrangian Density: $\mathcal{L}=\frac{1}{4\mu_0}(F_{\mu\nu}F^{\mu\nu})$
It corresponds to the following equations of motion (Lorentz Gauge): $\partial^\nu\partial_\nu A^\mu=0$
Through fourier transform, equation of motion of harmonic oscillator with momentum $\vec{k}$
Field is infinite collection of harmonic oscillators with infinite modes
Quantization of field is quantization of all harmonic oscillators using canonical commutation relations
Note: $a^\dagger$ is creation operator while a is annihilation operator

This allows for the following Fourier Expansion: $\vec{A}(x)=\frac{1}{\sqrt{V}}\sum_\lambda\sum_{\vec{k}} f(\vec{k})\,\hat{\varepsilon}_\lambda(\vec{k})\left[a^\dagger_\lambda(\vec{k})e^{-ikx}+a_\lambda(\vec{k})e^{ikx}\right]$

$V\equiv$ Volume of the box $\longrightarrow V^{-1/2}$ for normalization

N.B.: $x\equiv x^\mu$ and $k=k_\mu$ with $\eta=\mathrm{diag}(-,+,+,+)$

Comm. Relationships:

$[a_{\lambda_1},a_{\lambda_2}]=[a^\dagger_{\lambda_1},a^\dagger_{\lambda_2}]=0 \quad \forall\,\lambda_1,\lambda_2$

$[a_{\lambda_1}(\vec{k}_1),a^\dagger_{\lambda_2}(\vec{k}_2)]=\delta^{\lambda_1}_{\lambda_2}\,\delta^{k_1}_{k_2}$

$A(\vec{x},t)$ is a real (i.e. Hermitian) operator quantized in a box of side L and volume V (and thus normalised $V^{-1/2}$) $\Longrightarrow k_i=\frac{2\pi}{L}n_i$ where $n_i=0,\pm1,\pm2,\ldots$
The summation happens over all possible wavevectors $\vec{k}$ (i.e. over all possible modes). As the photon field can actually have two polarisations perpendicular to $\vec{k}$, there are two possible polarisations vectors $\hat{\varepsilon}_\lambda$ which satisfy $\vec{k}\cdot\hat{\varepsilon}_\lambda=0$. Therefore, to include all possibilities, a summation over all possible polarisations vectors is required (i.e. summation over $\lambda=1,2$). If $\vec{k}=k\hat{z}$, $\hat{\varepsilon}_1=\frac{1}{\sqrt{2}}(\hat{x}+i\hat{y})$ $\hat{\varepsilon}_2=\frac{1}{\sqrt{2}}(\hat{x}-i\hat{y})$

Classical Hamiltonian

From the Lagrangian one can derive the Hamiltonian: $H = \frac{1}{2}\varepsilon_0 \int d^3x \left[\vec{E}^2(x) + c^2\vec{B}(x)\right]$

$\hookrightarrow$ The Fields are: $\vec{E} = -\vec{\nabla}V - \partial_t \vec{A}$ $\quad \vec{B} = \vec{\nabla}\times\vec{A}$

By Coulomb gauge: $V = \vec{\nabla}\cdot\vec{A} = 0 \implies \vec{k}\cdot\hat{\varepsilon}_\lambda(\vec{k}) = 0$ Only Transverse Polarizations!

It follows that:

$$\vec{E} = -\partial_t\vec{A} = \frac{1}{\sqrt{V}}\sum_\lambda\sum_{\vec{k}} f(\vec{k})\hat{\varepsilon}_\lambda(\vec{k}) i\omega\left[a^\dagger_\lambda(\vec{k})e^{-ikx} - a_\lambda(\vec{k})e^{ikx}\right]$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{\sqrt{V}}\sum_\lambda\sum_{\vec{k}} f(\vec{k})\left\{\left[a^\dagger_\lambda(\vec{k})e^{-ikx} + a_\lambda(\vec{k})e^{ikx}\right]\left(\vec{\nabla}\times\hat{\varepsilon}_\lambda(\vec{k})\right) + \left[\vec{\nabla}\left(a^\dagger_\lambda(\vec{k})e^{-ikx} + a_\lambda(\vec{k})e^{ikx}\right)\right]\times\hat{\varepsilon}_\lambda(\vec{k})\right\} =$$

$$= \frac{1}{\sqrt{V}}\sum_\lambda\sum_{\vec{k}} f(\vec{k})\left[\vec{\nabla}\left(a^\dagger_\lambda(\vec{k})e^{-ikx} + a_\lambda(\vec{k})e^{ikx}\right)\right]\times\hat{\varepsilon}_\lambda(\vec{k}) = -\frac{1}{\sqrt{V}}\sum_\lambda\sum_{\vec{k}} f(\vec{k})\left[a^\dagger_\lambda(\vec{k})e^{-ikx} - a_\lambda(\vec{k})e^{ikx}\right]\left(i\vec{k}\times\hat{\varepsilon}_\lambda(\vec{k})\right)$$

Therefore:

$$E^2 = \vec{E}\cdot\vec{E}^* = \frac{1}{V^2}\sum_{k_1}\sum_{k_2}\sum_{\lambda_1}\sum_{\lambda_2} f(\vec{k}_1)f(\vec{k}_2)(-i\omega_1)(i\omega_2)\left[a^\dagger_{\lambda_1}(\vec{k}_1)e^{-ik_1x} - a_{\lambda_1}(\vec{k}_1)e^{ik_1x}\right]\left[a_{\lambda_2}(\vec{k}_2)e^{ik_2x} - a^\dagger_{\lambda_2}(\vec{k}_2)e^{-ik_2x}\right](\hat{\varepsilon}_{\lambda_1}\cdot\hat{\varepsilon}^*_{\lambda_2}) =$$

$$= \frac{1}{V^2}\sum_{k_1}\sum_{k_2}\sum_{\lambda_1}\sum_{\lambda_2} f(\vec{k}_1)f(\vec{k}_2)(\omega_1\omega_2)\left[a^\dagger_{\lambda_1}(\vec{k}_1)a_{\lambda_2}(\vec{k}_2)e^{-i(k_1-k_2)x} + a_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2)e^{i(k_1-k_2)x} - a^\dagger_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2)e^{-i(k_1+k_2)x} - a_{\lambda_1}(\vec{k}_1)a_{\lambda_2}(\vec{k}_2)e^{i(k_1+k_2)x}\right](\hat{\varepsilon}_{\lambda_1}\cdot\hat{\varepsilon}^*_{\lambda_2})$$

$$c^2B^2 = c\vec{B}\cdot c\vec{B}^* = \frac{c^2}{V^2}\sum_{k_1}\sum_{k_2}\sum_{\lambda_1}\sum_{\lambda_2} f(\vec{k}_1)f(\vec{k}_2)(k_1k_2)\left[a^\dagger_{\lambda_1}(\vec{k}_1)a_{\lambda_2}(\vec{k}_2)e^{-i(k_1-k_2)x} + a_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2)e^{i(k_1-k_2)x} - a^\dagger_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2)e^{-i(k_1+k_2)x} - a_{\lambda_1}(\vec{k}_1)a_{\lambda_2}(\vec{k}_2)e^{i(k_1+k_2)x}\right]\left[(\hat{k}_1\times\hat{\varepsilon}_1)\cdot(\hat{k}_2\times\hat{\varepsilon}_2)\right]$$

<u>The $\vec{E}$-term:</u>

$e^{i(\omega_2-\omega_1)t}$ $\quad e^{i(\omega_1+\omega_2)t}$

$$H_E = \frac{1}{2}\varepsilon_0\int d^3x\, E^2(\vec{x},t) = \frac{\varepsilon_0}{2V}\sum_{k_1}\sum_{k}\sum_{\lambda_1}\sum_{\lambda_2} f(\vec{k}_1)f(\vec{k}_2)(\omega_1\omega_2)\left[\left(a^\dagger_{\lambda_1}(\vec{k}_1)a_{\lambda_2}(\vec{k}_2) + a_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2)\right)\left(V\delta(\vec{k}_1-\vec{k}_2)\right) - \left(a^\dagger_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2) + a^\dagger_{\lambda_1}(\vec{k}_1)a^\dagger_{\lambda_2}(\vec{k}_2)\right)\left(V\delta(\vec{k}_1+\vec{k}_2)\right)\right](\hat{\varepsilon}_{\lambda_1}\cdot\hat{\varepsilon}^*_{\lambda_2}$$

$$= \frac{1}{2}\varepsilon_0\sum_k\sum_{\lambda_1}\sum_{\lambda_2} f(\vec{k})f(\vec{k})\,\omega\,\omega\left[\left(a^\dagger_{\lambda_1}(\vec{k})a_{\lambda_2}(\vec{k}) + a_{\lambda_1}(\vec{k})a^\dagger_{\lambda_2}(\vec{k})\right)\left(\hat{\varepsilon}_{\lambda_1}(\vec{k})\cdot\hat{\varepsilon}_{\lambda_2}(\vec{k})\right)\right] - f(\vec{k})f(-\vec{k})\omega_1\omega_2\left[\left(a^\dagger_{\lambda_1}(\vec{k})a^\dagger_{\lambda_2}(-\vec{k}) + a^\dagger_{\lambda_1}(\vec{k})a^\dagger_{\lambda_2}(-\vec{k})\right)\left(\hat{\varepsilon}_{\lambda_1}(\vec{k})\cdot\hat{\varepsilon}^*_{\lambda_2}(-\vec{k})\right)\right] =$$

$$= \frac{1}{2}\varepsilon_0\sum_k\sum_{\lambda_1}\sum_{\lambda_2} f(\vec{k})f(\vec{k})\omega^2\left[a^\dagger_{\lambda_1}(\vec{k})a_{\lambda_2}(\vec{k}) + a_{\lambda_1}(\vec{k})a^\dagger_{\lambda_2}(\vec{k})\right]\delta^{\lambda_1}_{\lambda_2} =$$

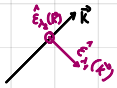
$$= \frac{1}{2}\varepsilon_0\sum_k\sum_\lambda f^2(\vec{k})\omega^2\left[a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k}) + a_\lambda(\vec{k})a^\dagger_k(\vec{k})\right] =$$

$$= \varepsilon_0\sum_k\sum_\lambda f^2(\vec{k})\omega^2\left[a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k}) + \frac{1}{2}\right] =$$

$$= \frac{1}{2}\sum_k\sum_\lambda \hbar\omega\left[a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k}) + \frac{1}{2}\right]$$

Sum of harmonic oscillators for all modes k! Thus $\varepsilon_0 f^2(\vec{k})\omega^2 = \frac{1}{2}\hbar\omega$

<u>Note:</u> $\int d^3x\, e^{i(k_1-k_2)x} = V\delta(\vec{k}_1-\vec{k}_2)$

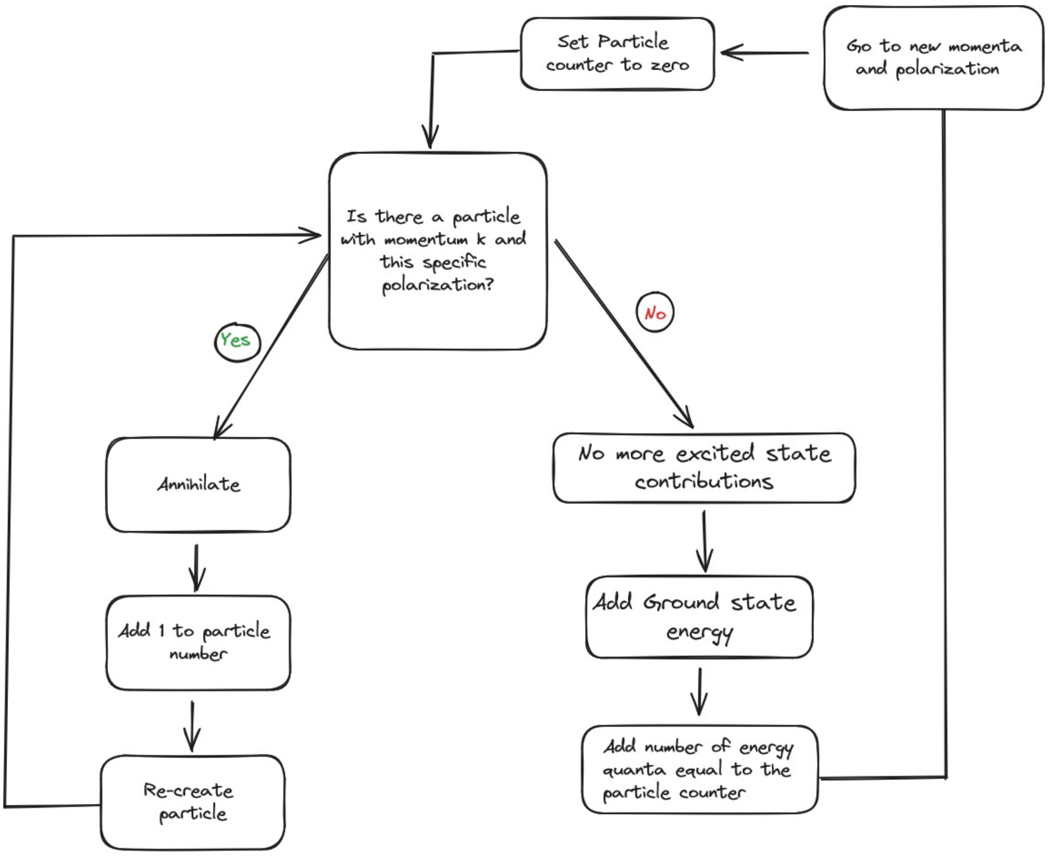


<u>The $\vec{B}$-Term:</u> Similarly to $\vec{E}$ $\quad H_B = \frac{1}{2}\varepsilon_0\int d^3x\, c^2B^2(\vec{x},t) = \frac{1}{2}\sum_k\sum_\lambda \hbar\omega\left[a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k}) + \frac{1}{2}\right]$

<u>The Hamiltonian is:</u> $H = \sum_k\sum_\lambda \hbar\omega\left[a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k}) + \frac{1}{2}\right] \implies$ Ground state energy is infinite

$(1/2)\,\hbar\omega_{k,\lambda}$ is ground state energy of oscillator $\implies$ infinite collection infinite energy

$\hbar\omega\, a^\dagger_\lambda(\vec{k})a_\lambda(\vec{k})$ counts particle by deleting and adding, adds energy for each particle counted



Lecture 25.03.2024

Casimir Effect

Hamiltonian: $H=\sum_{k,\lambda}\hbar\omega_{\vec{k}}(\underbrace{a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}}_{N\,\equiv\,\text{Counting operator}}+\frac{1}{2})$ with bosonic quantization relations $[a_{\vec{k}_1,\lambda_1},a^\dagger_{\vec{k}_2,\lambda_2}]=\delta^{k_2}_{k_1}\delta^{\lambda_2}_{\lambda_1}$

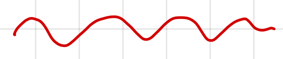
Vacuum state: $|0\rangle$ as it as zero quanta

n-photon state: $|n\rangle$ s.t. $a^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$ and $a|n\rangle=\sqrt{n}\,|n-1\rangle$

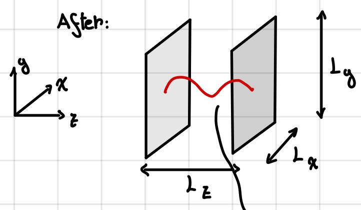
Note: We dropped momentum and polarization in state: $|n\rangle=|k_1,\lambda_1;\dots;k_n,\lambda_n\rangle$

Casimir Effect: Vacuum energy in a (in)finite region is infinite. By introducing two conducting plates we modify vacuum energy leading to finite calculable effective

Before:



After:



Standing wave blocked by plates

Due to plates, all oscillator states must satisfy boundary conditions

The effect must be a Pressure P i.e. Force per meter^2

$[P]=[F][m^2]=\frac{kg\cdot m/s^2}{m^2}=\frac{kg\cdot m^3/s^2}{m^4}=\frac{J\cdot m}{m^4}\Longrightarrow P=\beta\frac{\hbar c}{L^4}$

According to Casimir, $\beta\neq0$ but $\beta<0$ i.e. attractive force

Vector field: $\vec{A}_{\vec{k},\lambda}(\vec{r},t)\sim\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}}\hat{\varepsilon}_{\vec{k},\lambda}\Longrightarrow\vec{A}(\vec{r})=A_x(\vec{r})\hat{x}+A_y(\vec{r})\hat{y}+A_z(\vec{r})\hat{z}$

When $L_x=L_y=L$

$A_x(\vec{r})=\sqrt{8/V}\,\varepsilon_x\,\cos(k_x x)\sin(k_y y)\sin(k_z z)$ | $k_x=\ell\pi/L$ | Standing wave conditions in cavity

$A_y(\vec{r})=\sqrt{8/V}\,\varepsilon_y\,\sin\quad\cos\quad\sin$ | $k_y=m\pi/L$ | Impose Coulomb Gauge: $\vec{\nabla}\cdot\vec{A}=0\Longrightarrow\frac{\pi}{L}(\ell\varepsilon_x+m\varepsilon_y)+\frac{\pi}{L_z}n\varepsilon_z=0$

$A_z(\vec{r})=\sqrt{8/V}\,\varepsilon_z\,\sin\quad\sin\quad\cos$ | $k_z=n\pi/L_z$ | Two degrees of freedom if n,ℓ or m are $\neq0$. If one is zero, only one polarization

$\omega_{\ell mn}=k_{\ell mn}c=\pi c\left[(\ell^2+m^2)/L^2+n^2/L_z^2\right]^{1/2}$

Zero point energy inside the cavity

$$\sum'_{\ell,m,n}\underbrace{(2)}_{\text{only in case }\ell,m,n\neq0}\frac{1}{2}\hbar\omega_{\ell mn}=\sum'_{\ell mn}\pi\hbar c\left[\frac{\ell^2}{L^2}+\frac{m^2}{L^2}+\frac{n^2}{L_z^2}\right]^{1/2}$$

Now, take $L\ggg L_z=d\Longrightarrow$ replace sums over ℓ,m by integrals

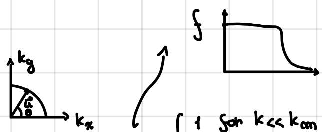
$$\sum_{\ell mn}\longmapsto\sum'_n\left(\frac{L}{\pi}\right)^2\int dk_x\,dk_y$$

$$E(d)=\frac{L^2}{\pi^2}\hbar c\sum'_n\int_0^\infty\int_0^\infty dk_x\,dk_y\left(k_x^2+k_y^2+\frac{n^2\pi^2}{d^2}\right)^{1/2}\Longrightarrow\infty$$ zero point energy in cavity (as in any other finite region in space)

Potential $U(d)=E(d)-E(\infty)$

Energy required to bring plates from a large distance to d

$$E(\infty)=(L^2/\pi^2)\hbar c\,\frac{d}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty dk_x\,dk_y\,dk_z\,(k_x^2+k_y^2+k_z^2)^{1/2}$$



polar coordinates in k_x,k_y plane $dk_x dk_y=u\,du\,d\theta$ θ from 0 to $\pi/2$ as

very high frequency waves do not see plates $\Longrightarrow$ insert cutoff function $f(k)=f\left(\sqrt{u^2+k_z^2}\right)\begin{cases}1 & \text{for } k\ll k_m\\ \approx0 & \text{for } k\gg k_m\sim1/a_0\end{cases}$ i.e. one over atomic size

$$U(d)=\frac{L^2\hbar c}{\pi^2}\frac{\pi}{2}\left[\sum'^{\infty}_{n=0}\int_0^\infty du\,u\left(u^2+\frac{n^2\pi^2}{d^2}\right)^{1/2}f\left(\sqrt{u^2+\frac{n^2\pi^2}{d^2}}\right)-\frac{d}{\pi}\int dk_z\int_0^\infty du\,u\,(u^2+k_z^2)^{1/2}f\left(\sqrt{u^2+k_z^2}\right)\right]$$

$$=\left(\frac{\pi^2\hbar c}{4d^3}\right)L^2\left[\frac{1}{2}F(0)+\sum_{n=1}^\infty F(n)-\int_0^\infty dk\,F(k)\right]$$

Euler-Maclaurin Formula: $\sum_{n=1}^\infty F(n)-\int_0^\infty dk\,F(k)=-\frac{1}{2}F(0)-\frac{4}{720}$ as $F'''(0)=-4$ while $F'(0)=F^{(i)}(0)$ for $i\geq4$

$$U(d)=-\frac{\pi^2}{720}\frac{\hbar c}{d^3}L^2$$

$$P(d)=-\frac{\pi^2}{240}\frac{\hbar c}{d^4}$$ finite Casimir force/area

$$P\sim-\frac{10^{-7}N}{[d(\mu m)]^4}$$